IN THE UNITED STATES PATENT AND TRADEMARK OFFICE

Applicants:	Uri Wilensky, Walter Stroup		
Title:	Distributed Agent Network Using Object Based Parallel Modeling Language to Dynamically Model Agent Activities		
Serial No.:	10/016,192	Filed:	December 12, 2001
Examiner:	David Silver	Group Art Unit:	2123
Docket No.:	045191.0001	Customer No.:	33438

Austin, Texas
November 10, 2006

Board of Patent Appeals and Interferences
United States Patent and Trademark Office
P.O. Box 1450
Alexandria, VA 22313-1450

APPEAL BRIEF UNDER 37 CFR § 41.37

Dear Sir:

Applicants submit this Appeal Brief pursuant to the Notice of Appeal filed in this case on September 11, 2006. The $250 fee for this Appeal Brief is being paid via the USPTO-EFS. The Board is also authorized to deduct any other amounts required for this appeal brief and to credit any amounts overpaid to Deposit Account No. 502264.

I. REAL PARTY IN INTEREST - 37 CFR § 41.37(c)(1)(i)

Because neither of the inventors has assigned the rights to this invention, the above-named inventors, Uri Wilensky and Walter Stroup, are the real parties in interest.

II. RELATED APPEALS AND INTERFERENCES - 37 CFR § 41.37(c)(1)(ii)

Based on information and belief, there are no appeals or interferences that could directly affect or be directly affected by or have a bearing on the decision by the Board of Patent Appeals and Interferences in the pending appeal. Pursuant to current Patent Office practice, Appendix "A" contains copies of all decisions rendered by a court or the Board in this "Related Appeals and Interferences" section, and is intentionally provided as an empty appendix.

III. <u>STATUS OF CLAIMS - 37 CFR § 41.37(c)(1)(iii)</u>

Claims 1-14 are pending in the application. Claims 1-14 stand rejected. The rejection of claims 1-14 is appealed. Appendix "B" contains the full set of pending claims.

IV. <u>STATUS OF AMENDMENTS - 37 CFR § 41.37(c)(1)(iv)</u>

On July 10, 2006, Applicant filed an Amendment and Response to Final Office Action requesting that claims 1-2 and 8-9 be amended. Applicants' response also included an evidentiary submission under 37 CFR § 1.116(e) to demonstrate the proper interpretation of the claim term "object-based parallel modeling language." In an Advisory Action dated August 4, 2006, the Examiner declined to enter the requested amendments.

V. <u>SUMMARY OF CLAIMED SUBJECT MATTER - 37 CFR § 41.37(c)(1)(v)</u>

The subject matter defined in independent claim 1 may be understood with reference to the example embodiment depicted in Figure 1 which depicts a modeling device for simulating complex dynamic systems (100). As recited, the complex dynamic system is embodied in the interaction of a plurality of remote agents (e.g., calculator devices 111, 113, 115, 117 and client machines 91, 93, 98), and is simulated at the central server computing device (109).

Each remote agent (*e.g.*, 111, 113) includes logic to receive input data. *See, e.g.*, Application, p. 11, ¶ 26 ("…a remote computer device detects some form of input at the device at step 210 (such as numeric keypad entry or other sensory device inputs)…"). Each remote agent (*e.g.*, 111, 113) also includes object control node information (114) corresponding to the performance of the remote agent and the relationship of the remote agent to the simulation. *See, e.g.*, Figure 1 (object control node 114). In addition, each remote agent includes control instructions to convert the input data into the control node information. *See, e.g.*, Application, p. 9, ¶ 24 ("…, a selected embodiment of the present invention enables the remote devices to efficiently and readily implement <u>individualized control input for each object in the form of strategies or rules</u>, and then to readily simulate the combined effect of the various inputs, strategies and rules from the distributed objects into a single simulation.") (emphasis added). An example of the "object control node information" and "control instructions" is described in the Application:

> When a remote computer device detects some form of input at the device at step 210 (such as numeric keypad entry or other sensory device inputs), <u>the remote device detects</u>

> the input and then transmits to the network 80 or centralized server 109 a flag indicating the "kind" and "content" of the information that has been detected. It will be appreciated that additional or alternative information can be transmitted upon detection of input at a remote device, such as a time stamp indication or some other programming component characterizing the detected input. In the example discussed herein of a simulation involving the positioning of multiple objects within a given space, the detected input could consist of a position movement indicator (such as an up, down, left or right signal) and/or a rule for controlling the motion of the object (such as an instruction to always move two spaces to the right whenever the object is co-positioned with another object in the physical space). These simplified examples will illustrate the functionality of the present invention to those skilled in the art concerning step 210.

See, *e.g.*, Application, p. 11, ¶ 26 (emphasis added). Finally, each remote agent includes logic for transmitting the object control node information and the control instructions to a centralized server computing device. *See*, *e.g.*, Application, ¶¶ 26-27 ("…the remote device detects and transmits information at step 210 to the centralized network or server…").

At the centralized server computing device (109), an object-based parallel modeling language component (108) is provided for collecting object control node information and control instructions from each of the plurality of remote agents. The object-based parallel modeling language component (108) also coordinates the interaction of the remote agents based upon the collected object control node information and control instructions. An example implementation is described in the Application:

> At the central server 109, the inputs from the various remote devices are collected and aggregated together using modeling, analysis and display tools that together are used to simulate in real time the complex interplay between the individual nodes or objects controlled by the remote devices. By providing an object-based parallel modeling language 108 at the server 109, users at the remote "nodes" can readily encode individualized strategies as rules which the system can then run independently of the other nodes, while simultaneously determining the resulting interplay between the various nodes and broadcasting the result to all remote devices.

See, *e.g.*, Application, p. 8, ¶ 22. The server computing device (109) also includes logic for transmitting the interactive simulation information based upon the interaction of the remote agents to the plurality of remote agents. *See*, *e.g.*, Application, p. 12, ¶ 29 ("Once the centralized simulation server determines for a particular object string that there is an effect that results from a co-positioning at step 220, that effect would be broadcast to all of the remote devices at step 225 to convey information about the result, and then the server would continue to process in sequence the remaining object strings.")

To comply with 37 CFR § 41.37(c)(1)(v), a color-coded comparison of independent claim 1 (including reference characters) and the relevant portion of Figure 1 is set forth below:

1. A modeling device for a simulation of complex dynamic systems, comprising:

a plurality of remote agents (e.g., 111, 113, 115, 117), each remote agent comprising:

logic to receive input data;

object control node information (e.g., 114) corresponding to performance of the remote agent and a relationship of the remote agent to the simulation;

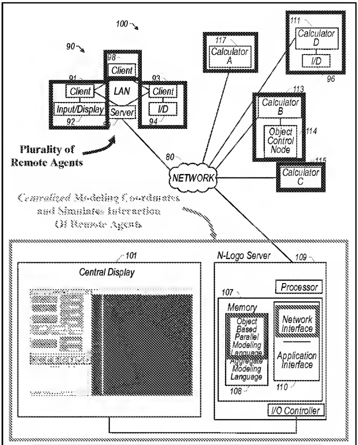
control instructions to convert the input data into the control node information; and

logic to transmit the object control node information and the control instructions to a server computing device; and

the server computing device (109), comprising:

an object-based parallel modeling language component (108) that collects object control node information and control instructions corresponding to each of the remote agents of the plurality of remote agents and coordinates the interaction of the remote agents based upon the collected object control node information and control instructions; and

logic (110) to transmit interactive simulation information based upon the interaction of the remote agents to the plurality of remote agents.



Wilensky et al. Application, Figure 1

In further compliance with 37 CFR § 41.37(c)(1)(v), a color-coded comparison of selected Figures from the application and each of the pending independent claims is attached at Appendix "C" to provide a concise explanation of the subject matter defined in each independent claim. The subject matter of the independent claims is set forth in the specification at page 4, line 5 to page 27, line 10.

VI. GROUNDS OF REJECTION TO BE REVIEWED ON APPEAL - 37 CFR § 41.37(c)(1)(vi)

In the Final Office Action dated May 10, 2006, the Examiner objected to claim 8 for an informality because the phrase "'the coordination' should be changed to 'a coordination.'" The Examiner also rejected claims 1-14 under 35 U.S.C. § 102(b) as being anticipated by U.S. Patent

No. 5,466,200 to Ulrich et al. (hereinafter "Ulrich"). In this appeal, Applicants appeal the following grounds of rejection:

A. The objection to claim 8 for the informality noted above and the Examiner's refusal to withdraw the objection when Applicants made the required correction in the Amendment After Final; and

B. The rejection of 1-14 under 35 U.S.C. § 102(b) as being anticipated by Ulrich.

VII. ARGUMENT - 37 CFR § 41.37(c)(1)(vii)

Each of pending claims 1-14 requires, with varying degrees of specificity, a server computing device to collect remote agent inputs and coordinate the interaction of the remote agent inputs to efficiently simulate a complex system of remote and independent inputs, thereby generating interactive simulation information based upon the interaction of the remote agents. *See, e.g.*, claim 1. In particular, the server computing device collects information from the remotes agents, and then uses this information to coordinate the interaction of the remote agents, thereby generating interactive simulation information that is transmitted to the remote agents. *Id.* As explained more fully below, the cited Ulrich reference fails to disclose a *centralized* server computing device that *centrally coordinates and simulates* the interaction of remote agents. To the contrary, the cited Ulrich reference discloses a distributed network of remote exercise machines, where each remote exercise machine has its own simulator.

A. The Objection to Claim 8 Should Be Withdrawn

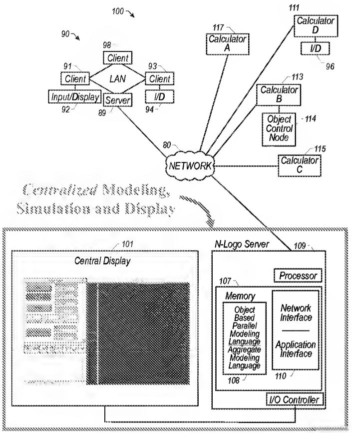
Applicants appeal the objection of claim 8 for the informality that the claim term "'the coordination' should be changed to 'a coordination.'" *See*, Final Office Action, p. 5 (May 10, 2006). This objection was raised *for the first time* in the Final Office Action, and the Examiner stated that appropriate correction was required. In particular, the Examiner's Final Office Action required Applicants to correct claim 8 so that the reference to "the coordination" was changed to "a coordination." In response, Applicants submitted an amendment to claim 8 in the Amendment and Response to Final Office Action dated July 10, 2006 which met the Examiner's requirement on this point. However, without any explanation, the Examiner refused to enter the amendment to claim 8. *See*, Advisory Action (August 4, 2006). Applicants submit that the refusal to enter the amendment was an arbitrary decision that was an abuse of any discretion that the Examiner had on this issue. Accordingly, Applicants respectfully request that the refusal to enter the requirement amendment be withdrawn and that the objection to claim 8 be withdrawn.

B. Claims 1-14 Are Not Anticipated By Ulrich

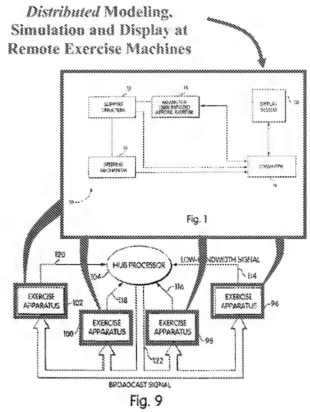
As explained more fully below, the rejection of claims 1-14 over Ulrich was erroneous and should be withdrawn because Ulrich's disclosure of a network of ***remote*** exercise machine simulators (each of which generates an interactive simulated environment) does not anticipate the present invention's **centralized server computing device** to collect remote agent inputs and coordinate the interaction of the remote agent inputs to efficiently simulate a complex system of remote and independent inputs.

1. Overview of Differences Between Claimed Invention and Ulrich

Applicants respectfully submit that the rejection of claims 1-14 is based on a misapplication of the claims to the Ulrich Patent. Generally speaking, the misapplication occurs when the **centralized** aspects of the claims (which variously recite a centralized server-based system for modeling, simulating, authoring, and/or displaying a complex and interactive dynamic system) are applied to the distributed aspects of Ulrich (which discloses a network of computerized remote exercise machines, each of which generates an interactive simulated environment using distributed database techniques). **In short, Ulrich distributes the simulation functionality at each remote exercise machine, while the present invention provides a centralized simulator.** This difference is graphically illustrated below with selected annotated figures from the present application (Figure 1) and the Ulrich Patent (Figures 1 and 9):



Wilensky et al. Application, Figure 1



Ulrich Patent, Figures 1 and 9

As these figures show, Applicants have centrally located the modeling, simulation and display aspects in the central server (e.g., the N-Logo Server), while **Ulrich has distributed these aspects in each of the remote exercise machines, not in the hub processor.** In particular, Ulrich discloses a network of computerized **remote exercise machines, each of which includes a "processor which generates an interactive simulated environment" using a shared "environmental database [that] is stored and executed on each machine."** Ulrich Patent, col. 2, lines 26-28; col. 8, lines 28-29; and col. 8, liens 34-37. Ulrich repeatedly discloses that the simulation is executed at the ***remote*** exercise machines. For example, when describing the *remote* exercise apparatus depicted in Figure 1, Ulrich teaches that the "interactive simulated environment is generated by a processor 18, such as a computer, and displayed on a display system 20." *See*, Ulrich Patent, col. 4, lines 9-11.

In this respect, Ulrich's distributed simulation approach was expressly distinguished by the Applicants in the application with the statement that:

> By centrally locating the modeling, analysis and display tools at the server 109, the aggregated results of the individual objects' behavior (controlled by the remote device inputs) can be efficiently simulated, thereby avoiding the complexity and time delays associated with distributing such functionality amongst the remote devices.

Application, p. 8 (paragraph 23) (emphasis added). Having specifically structured the claimed invention to recite that the server computing device includes a modeling language component that collects remote inputs and coordinates the interaction of the remote agents in a centralized simulation, thereby generating interactive simulation information, Applicants respectfully submit that Ulrich fails to disclose such a centralized simulation. *See, e.g.*, claim 1.

The Examiner appears to acknowledge this argument, but responds with the assertion that Ulrich discloses (at col. 11, lines 31-64) that the hub processor 104 "coordinates" the interaction of remote agents (exercise machines) by disseminating control information and instructions received from one remote agent to the other agents, thereby meeting the centralized modeling requirement. *See*, Final Office Action, pp. 2-3 (paragraph 8). The Examiner then asserts that Applicants have not claimed a central server that performs modeling of the remote agents. *See*, Final Office Action, p. 3 ("Additionally, it seems the Applicants are arguing that the central server of their claimed invention performs a modeling of the remote agents. This however is not claimed."). Applicants must strenuously disagree. It should first be noted that the language of claim 1 explicitly recites a "modeling device for a simulation of complex dynamic systems." *See above*, claim 1, line 1 (emphasis added). More importantly, claim 1 *explicitly* recites that the "server computing device" includes "an object-based modeling language component." *See*, claim 1, line 12 (emphasis added). In addition, claim 1 requires that "interactive simulation information" be transmitted by the server computing device.

In view of Ulrich's failure to meet these requirements, Applicants respectfully submit that the Examiner has incorrectly applied the legal requirements for establishing anticipation by the Ulrich reference. To establish a *prima facie* case of anticipation, the Examiner has the burden of pointing out where each and every element of the claimed invention, arranged as required by the claims, are found in the Ulrich reference, either expressly or under the principles of inherency. *See generally*, In re King, 801 F.2d 1324, 1326, 231 USPQ 136, 138 (Fed. Cir. 1986); Lindemann Maschinenfabrik GMBH v. American Hoist and Derrick, 730 F.2d 1452, 1458, 221 USPQ 481, 485 (Fed. Cir. 1984). Since Ulrich discloses a distributed simulation system, Applicants submit that at least the centralized simulation requirement of the claims is

missing. Accordingly, Applicants respectfully request that the anticipation rejection of claims 1-14 be withdrawn and that the claims be allowed.

2. Ulrich's Hub Server Does Not Include A "Modeling Language Component," Much Less An "Object-Based Parallel Modeling Language Component"

Another deficiency is that Ulrich entirely fails to teach or disclose the requirement in claims 1-7 of a server-side "object-based parallel modeling language component" that coordinates remote inputs at the server to efficiently simulate a complex system of remote and independent inputs. Indeed, there is no reference in Ulrich to any "modeling language," whether "object-based," "parallel" or otherwise! This is confirmed by a careful review of the Ulrich passages cited by the Examiner -- "col.: 8 lines: 53-55, Figure 8 and 9 item 104; Figure 11 object-based parallel modeling language component ... item 138/136" -- which do meet the OBPML requirement. For example, the Ulrich passage referenced by the Examiner at column 8 ("This dual use of the phone signal is possible due to the relatively low-bandwidth of communication required for the shared parameters (e.g., position, direction).") says nothing about the OBPML component. As for the reference to items 138/139 in Figure 11, this figure illustrates the flow chart process for software executing on "<u>each networked exercise apparatus</u>" and the items 138 and 139 refer to the "interactive mode" and "download mode" steps taken by the <u>remote exercise machine</u> takes when networked to other computers. Again, no reference to any OBPML component at the server. Lastly, the reference to the hub processor 104 shown in Figures 8 and 9 is similarly unavailing since Ulrich describes the function of the hub processor 104 as a database distributor that is "responsible for limiting the information directed to each apparatus in the large-scale direct network of FIG. 8. The hub 104 can ensure, for example, that each apparatus only gets (parameter) updates about other users in the same general area of the simulated environment." Ulrich Patent, col. 9, lines 5-10. This database distribution function of the hub processor 104 is confirmed by Ulrich's description of the hub processor block diagram (Figure 12) and process flow flowchart (Figure 13), which confirm that Ulrich's hub processor 104 performs database update and distribution functions without ever mentioning or suggesting any modeling function, much less any "object-based parallel modeling language component." Ulrich's use of the hub processor to distribute database information does not meet the claim requirement of a server-side modeling language component for coordinating the interaction of remote agents.

Rather than accepting these clear teachings from Ulrich, the Examiner asserts that the "object-based modeling component" is disclosed by Ulrich's references to an "object database" (col. 11, lines 46-49) and some "persona selection" software (col. 10, lines 2-7). Final Office Action, pp. 3-4 (paragraph 10). The Examiner also asserts for the first time in the Final Office Action that object-based parallel computer modeling languages are well known in the art, and that in the absence of an explicit definition in the application, the term will be given its ordinary meaning in the art, though the Examiner never states what this meaning is. *Id.* (paragraph 12). As explained below, these assertions are not consistent with the plain meaning of the claim language or the usage in the present application.

While Applicants agree that the ordinary and customary meaning of the claim term should apply, Applicants respectfully submit that the Examiner has not used the ordinary meaning of an "object-based" modeling language, which is properly understood to refer to a modeling language that acts on each "object" as a self-contained unit that has its own internal state and can be given its own local procedures and rules of interaction.[1] *See, e.g.*, Microsoft Computer Dictionary, p. 338 (3d ed. 1997) ("**object-oriented programming** ... A programming paradigm in which a program is viewed as a collection of discrete objects that are self-contained collections of data structures and routines that interact with other objects.") (attached as Exhibit B in Appendix D).[2] Examples of such object-based languages include, but are not limited to, Simula, Smalltalk, C++, Objective-C, Eiffel, Python, Java, C#, Visual Basic.NET and REALbasic.

More importantly, Applicants' interpretation complies with the requirement that, during patent examination, the pending claims must be "given their broadest reasonable interpretation consistent with the specification." *See*, MPEP, § 2111 (citing *In re Hyatt*, 211 F.3d 1367, 1372,

[1] The dictionary and other extrinsic evidence relating to the proper interpretation of the claim term "object-based parallel modeling language" attached at Appendix D was submitted under 37 CFR § 1.116(e) as part of Applicants Amendment and Response After Final Rejection. Though Applicants submit that the information in the enclosed references reflects what was known to those of ordinary skill in the art, these materials are nevertheless being presented out of an abundance of caution. In support of the Rule 1.116(e) submission, Applicants stated that this information is necessary to demonstrate to the Examiner the proper interpretation of the "object-based parallel modeling language" term. Applicants also stated that this information was not earlier presented because the Examiner first asserted to be applying the "ordinary" (but unspecified) meaning of this term in the final Office Action.

[2] It should be noted that the Microsoft Dictionary definitions differentiate between an "object-oriented database" and "object-oriented programming."

54 USPQ2d 1664, 1667 (Fed. Cir. 2000)) (emphasis added). In this respect, Applicants' specification states that "object-based parallel computer modeling languages (OBPML), such as StarLogo and StarLogoT (Resnick, 1994; Wilensky, 1995; 1997b), have previously been developed." Application, paragraph 9. As stated in the attached copy of the cited "Wilensky, 1995" article, the plain meaning of an object-based modeling language refers to a programming language that acts on an object that has its "own local state and can be given its own local procedures and rules of interaction." *See*, U. Wilensky, "Paradox, Programming and Learning Probability: A Case Study In A Connected Mathematics Framework," Journal of Mathematical Behavior, Vol. 14, No. 2, fn 9 and associated text (1995) ("Each of the turtles (or agents) has its own local state and can be given its own local procedures and rules of interaction.... These 'object-oriented' features of the language make StarLogo a more accessible environment for modeling. In contrast to other modeling environments, such as STELLA (Richmond & Peterson, 1990), which model with aggregate quantities and flows, StarLogo is 'object' based - thus facilitating concrete interactions with the basic units of the model.") (attached as Exhibit C in Appendix D). Applicants' interpretation is also consistent with the specification's statement that:

> complex systems can be modeled and analyzed using the object-based parallel modeling language (OBPML) aspect of the present invention. OBPMLs afford a probabilistic and statistical approach to modeling which, in a distributed network embodiment of the present invention, can provide an improved method and system for simulating complex and dynamic systems.

Application, paragraph 12. These passages from the Applicants' specification confirm that an "object-based modeling language" refers to a modeling language that acts on self-contained objects, each of which has its own internal state. *See also*, U. Wilensky, "What Is Normal Anyway? Therapy For Epistemological Anxiety," Educational Studies in Mathematics, Vol. 33, No. 2, pp. 171-202, § 5.2 (1997) ("'Object-based' means that each agent is self-contained: it has its own internal state and communicates with other agents primarily by local channels - agents don't do much action at a distance. The computer language Logo had a single such object - the 'turtle'.... Object-based parallel modeling languages such as StarLogo afford greater identification with their objects, and thus, in contrast to more procedural languages, foster syntonic learning of emergent phenomena.") (attached as Exhibit D in Appendix D).

Based on the correct interpretation of the "object-based parallel modeling language component" claim term, Applicants respectfully submit that claims 1-7 are not anticipated by Ulrich's description of the hub processor 104. A careful reading of the cited Ulrich passages (col. 11, lines 31-64) and the associated Figure 13 confirms that Ulrich is describing the distributed database techniques used by the hub server 104 to distribute and update databases in response to requests from the remote exercise machines. Indeed, there is no reference in Ulrich to any "modeling language" at the hub server, whether "object-based," "parallel" or otherwise! Accordingly, Applicants respectfully request that the anticipation rejection of claims 1-7 be withdrawn and that the claims be allowed.

3. Ulrich's Hub Server Does Not Include a Central Graphical Display

Yet another deficiency is that Ulrich entirely fails to disclose a server computing device that includes "modeling tools," "analysis tools" and "display tools." With respect to the display limitation, a number of the claims expressly recite that the central server includes a central control panel having a graphical display for viewing the simulation information. *See*, claims 2, 5-7 and 12-14. In rejecting these claims, the Examiner relied on Ulrich's description of the display at the remote exercise machine (Figure 2A, item 35). *See*, <u>Final Office Action</u>, p. 3 (paragraph 9). However, the <u>only</u> displays described in Ulrich area those associated with the remote exercise apparatus. *See*, Ulrich Patent, Figures 1 and 10 (display system 20 in the depicted exercise apparatus), Figure 2A (display 35 in the depicted exercise cycle), Figure 2B (monitors 44, 46 in the depicted exercise cycle), and Figures 3 and 11 (visual display step 62 in the depicted process flow for the exercise cycle computer 32). In contrast, there is no suggestion or disclosure by Ulrich that the hub processor 104 includes a display.

Once the centralized simulation and display aspects are taken into account, it becomes clear that Ulrich's disclosure of a network of <u>remote</u> exercise machine simulators (each of which generates and displays an interactive simulated environment) does not anticipate the present invention's use of a central server computing device to collect remote agent inputs, simulate the interaction of the remote agent inputs display the resulting simulation. These differences alone are sufficient to differentiate the Ulrich disclosure as explained above, though there are other differences that flow therefrom, including the server-based modeling and analysis requirements of claims 2 and 9, the server-transmitted interactive simulation information of claims 3 and 10, and the server-based display requirements of claims 2, 5-7 and 12-14, none of which are

disclosed by Ulrich. Accordingly, Applicants respectfully request that the anticipation rejection of claims 1-14 be withdrawn and that the claims be allowed.

VIII. CLAIMS APPENDIX - 37 CFR § 41.37(c)(1)(viii)

A copy of the pending claims involved in the appeal is attached as Appendix "B."

IX. EVIDENCE APPENDIX - 37 CFR § 41.37(c)(1)(ix)

Applicants are not submitting any evidence pursuant to 37 CFR §§ 1.130, 1.131, or 1.132 of this title, and based on information and belief, the Examiner is not submitting any evidence that Applicant will be relying upon (assuming that the office actions and cited references applied by the Examiner in the Final Office Action for this case do not qualify as "evidence entered by the examiner"). As for the above-referenced dictionary and other extrinsic evidence relating to the proper interpretation of the claim term "object-based parallel modeling language" (attached at Exhibits B-D in Appendix D), this evidence was submitted under 37 CFR § 1.116(e) as part of Applicants' Amendment and Response After Final Rejection. Though Applicants submit that the information in the enclosed references reflects what was known to those of ordinary skill in the art, these materials are nevertheless being presented out of an abundance of caution. In support of the Rule 1.116(e) submission, Applicants stated that this information is necessary to demonstrate to the Examiner the proper interpretation of the "object-based parallel modeling language" term. Applicants also stated that this information was not earlier presented because the Examiner first asserted to be applying the "ordinary" (but unspecified) meaning of this term in the Final Office Action.

While the Examiner apparently declined to consider the Rule 116(e) information because it was not submitted in an Information Disclosure Statement (Advisory Action, p.2), Applicants submit that there is no such requirement for Rule 116(e) submissions, and that this properly submitted evidence should be fully considered here. Pursuant to current Patent Office practice, Appendix "D" contains copies of all evidence identified in this "Evidence Appendix" section.

X. RELATED PROCEEDINGS APPENDIX - 37 CFR § 41.37(c)(1)(x)

There are no related proceedings.

XI. CONCLUSION

For the reasons set forth above, Applicants respectfully submit that the cited Ulrich reference does not anticipate the present invention's use of a central server computing device to collect remote agent inputs, simulate the interaction of the remote agent inputs and display the resulting simulation. To the contrary, Ulrich typifies the prior art with its disclosure of remote simulators at each remote exercise machine. Accordingly, Applicants respectfully submit that rejection of pending claims 1-14 is unfounded, and requests that the rejection of claims 1-14 be reversed.

ELECTRONICALLY FILED November 10, 2006

Respectfully submitted,

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APPENDIX A - RELATED APPEALS AND INTERFERENCES

None

APPENDIX B

1. A modeling device for a simulation of complex dynamic systems, comprising:

a plurality of remote agents, each remote agent comprising:

logic to receive input data;

object control node information corresponding to performance of the remote agent and a relationship of the remote agent to the simulation;

control instructions to convert the input data into the control node information; and

logic to transmit the object control node information and the control instructions to a server computing device; and

the server computing device, comprising:

an object-based parallel modeling language component that collects object control node information and control instructions corresponding to each of the remote agents of the plurality of remote agents and coordinates the interaction of the remote agents based upon the collected object control node information and control instructions; and

logic to transmit interactive simulation information based upon the interaction of the remote agents to the plurality of remote agents.

2. The modeling device of claim 1, the server computing device further comprising:

modeling tools;

analysis tools; and

display tools.

3. The modeling device of claim 1, wherein the interactive simulation information is transmitted to a particular remote agent only if the simulation information of the particular remote agent is impacted by control node information and control instructions of a second remote agent.

4. The modeling device of claim 1, wherein the input information comprises:

input data; and

control instructions corresponding to the remote agent.

5. The modeling device of claim 1, the server further comprising:

a central control panel comprising:

a graphical display for viewing the simulation information.

6. The modeling device of claim 5, wherein the graphical display also displays input information and status data for a selected remote agent of the plurality of remote agents.

7. The modeling device of claim 5, the central control panel further comprising:

a plurality of user input devices for providing direct interaction with the object-based parallel modeling language component by enabling a user to input information and control instructions, both corresponding to a selected remote device.

8. A method of producing a coordinated and interactive simulation of a dynamic system, comprising the steps of:

defining a set of remote agents, wherein each remote agent performs the steps of:

receiving input data;

transmitting the input data and control instructions relating to a corresponding remote agent of the set of remote agents to a server computing device; and

collecting the input data and control instructions from each of the remote agents of the plurality of remote agents at the server computing device;

coordinating the interaction of the remote agents at the server computing device based upon the input data and the control instructions, each set of control instructions corresponding to the set of control instructions of each remote agent of the plurality of remote agents; and

transmitting interactive simulation information based upon the coordination of the interaction of the remote agents from the server computing device to the plurality of remote agents.

9. The simulation method of claim 8, the coordinating step comprising the steps of:

analyzing the input data corresponding to a particular remote agent based upon control instructions corresponding to the particular remote agent;

modeling the interactive simulation information based upon an interaction between the analyzed input data from the remote agents; and

displaying a simulation based upon the interactive simulation information.

10. The simulation method of claim 8, wherein the interactive simulation information is transmitted to a particular remote agent only if the simulation information for the particular remote agent is impacted by control node information and control instructions of a second remote agent.

11. The simulation method of claim 8, further comprising the step of:

defining sets of control instructions, each set of control instructions corresponding to a remote agent of the plurality of remote agents; and

input to each particular remote agent the set of control instructions corresponding to the particular remote agent.

12. The simulation method of claim 8, further comprising the step of:

displaying on a central control panel coupled to the server computing device a graphical display of the interactive simulation information.

13. The simulation method of claim 12, further comprising the step of:

displaying on the central control panel input information and status data for a selected remote agent of the plurality of remote agents.

14. The simulation method of claim 12, further comprising the step of:

entering input information and control instructions, both corresponding to a selected remote device, at the server computing device.

APPENDIX C

1. A modeling device for a simulation of complex dynamic systems, comprising:

a plurality of remote agents, each remote agent comprising:

logic to receive input data;

object control node information corresponding to performance of the remote agent and a relationship of the remote agent to the simulation;

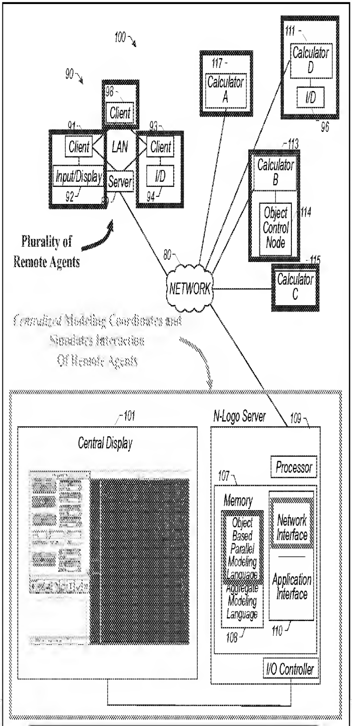
control instructions to convert the input data into the control node information; and

logic to transmit the object control node information and the control instructions to a server computing device; and

the server computing device, comprising:

an object-based parallel modeling language component that collects object control node information and control instructions corresponding to each of the remote agents of the plurality of remote agents and coordinates the interaction of the remote agents based upon the collected object control node information and control instructions; and

logic to transmit interactive simulation information based upon the interaction of the remote agents to the plurality of remote agents.



Wilensky et al. Application, Figure 1

8. A method of producing a coordinated and interactive simulation of a dynamic system, comprising the steps of:

defining a set of remote agents, wherein each remote agent performs the steps of:

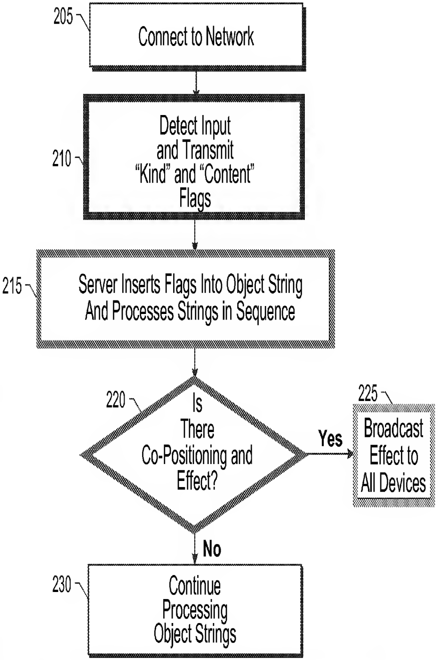
receiving input data;

transmitting the input data and control instructions relating to a corresponding remote agent of the set of remote agents to a server computing device; and

collecting the input data and control instructions from each of the remote agents of the plurality of remote agents at the server computing device;

coordinating the interaction of the remote agents at the server computing device based upon the input data and the control instructions, each set of control instructions corresponding to the set of control instructions of each remote agent of the plurality of remote agents; and

transmitting interactive simulation information based upon the coordination of the interaction of the remote agents from the server computing device to the plurality of remote agents.



Wilensky et al. Application, Figure 2

APPENDIX D - EVIDENCE APPENDIX

EXHIBIT B

Microsoft Press Computer Dictionary

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system or new software is developed. *See also* object (definition 2), object-oriented design, object-oriented programming.

object-oriented database \ob`jekt-ōr-ē-ent-ad dā´ta-bās\ *n.* A flexible database that supports the use of abstract data types, objects, and classes and that can store a wide range of data, often including sound, video, and graphics, in addition to text and numbers. Some object-oriented databases allow data retrieval procedures and rules for processing data to be stored along with the data or in place of the data. This allows the data to be stored in areas other than in the physical database, which is often desirable when the data files are large, such as those for video files. *Acronym:* OODB (O`O-D-B´). *See also* abstract data type, class, object (definition 2). *Compare* relational database.

object-oriented design \ob`jekt-ōr-ē-en-tad da-zīn´\ *n.* A modular approach to creating a software product or computer system, in which the modules (objects) can be easily and affordably adapted to meet new needs. Object-oriented design generally comes after object-oriented analysis of the product or system and before any actual programming. *See also* object (definition 2), object-oriented analysis.

object-oriented graphics \ob`jekt-ōr-ē-en-tad graf´iks\ *n.* Computer graphics that are based on the use of graphics primitives, such as lines, curves, circles, and squares. Object-oriented graphics, used in applications such as computer-aided design and drawing and illustration programs, describe an image mathematically as a set of instructions for creating the objects in the image. This approach contrasts with the use of bit-mapped graphics, in which a graphic is represented as a group of black-and-white or colored dots arranged in a certain pattern. Object-oriented graphics enable the user to manipulate objects as units. Because objects are described mathematically, object-oriented graphics can be layered, rotated, and magnified relatively easily. *Also called* structured graphics. *See also* graphics primitive. *Compare* bit-mapped graphics, paint program.

object-oriented interface \ob`jekt-ōr-ē-en-tad in´tar-fās\ *n.* A user interface in which elements of the system are represented by visible screen entities, such as icons, that are used to manipulate the system elements. Object-oriented display interfaces do not necessarily imply any relation to object-oriented programming. *See also* object-oriented graphics.

object-oriented operating system \ob`jekt-ōr-ē-en-tad op´ar-ā-tēng si`-stam\ *n.* An operating system based on objects and designed in a way that facilitates software development by third parties, using an object-oriented design. *See also* object (definition 2), object-oriented design.

object-oriented programming \ob`jekt-ōr-ē-en-tad prō´gram`ēng\ *n.* A programming paradigm in which a program is viewed as a collection of discrete objects that are self-contained collections of data structures and routines that interact with other objects. *Acronym:* OOP (ōōp, O`O-P´). *See also* C++, object (definition 2), Objective-C.

object-relational server \ob`jekt-ra-lā`sha-nal sar´var\ *n.* A database server that supports object-oriented management of complex data types in a relational database. *See also* database server, relational database.

object request broker \ob`jekt ra-kwest` brō-kar\ *n.* *See* ORB.

object wrapper \ob´jekt rap`ar\ *n.* In object-oriented applications, a means of encapsulating a set of services provided by a non-object-oriented application so that the encapsulated services can be treated as an object. *See also* object (definition 2).

oblique \ō-blēk´\ *adj.* Describing a style of text created by slanting a roman font to simulate italics when a true italic font isn't available on the computer or printer. *See also* font, italic, roman.

OC3 \O`C-thrē´\ *n.* Short for optical carrier 3. One of several optical signal circuits used in the SONET high-speed fiber-optic data transmission system. OC3 carries a signal of 155.52 Mbps, the minimum transmission speed for which SONET and the European standard, SDH, are fully interoperable. *See also* SONET.

OCR \O`C-R´\ *n.* *See* optical character recognition.

octal \ok´tal\ *n.* The base-8 number system consisting of the digits 0 through 7, from the Latin *octo,* meaning "eight." The octal system is used in programming as a compact means of representing binary numbers. See Appendix E. *See also* base (definition 2).

EXHIBIT C

Paradox, Programming and Learning Probability: A Case Study in a Connected Mathematics Framework

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ABSTRACT

Formal methods abound in the teaching of probability and statistics. In the Connected Probability project, we explore ways for learners to develop their intuitive conceptions of core probabilistic concepts. This paper presents a case study of a learner engaged with a probability paradox. Through engaging with this paradoxical problem, she develops stronger intuitions about notions of randomness and distribution and the connections between them. The case illustrates a Connected Mathematics approach: that primary obstacles to learning probability are conceptual and epistemological, that engagement with paradox can be a powerful means of motivating learners to overcome these obstacles, that overcoming these obstacles involves learners making mathematics - not learning a "received" mathematics and that, through programming computational models, learners can more powerfully express and refine their mathematical understandings.

Introduction | Conclusion | References

1. Introduction

The disciplines of probability and statistics have fundamentally changed the way we do science and the way we think about our world. Many scholars have argued (e.g., Cohen, 1990; Gigerenzer, 1990; Hacking, 1990) that a probabilistic revolution has occurred in our century and that notions of randomness and uncertainty have opened up whole new areas of mathematics and science. This has released a ground swell of interest in subjects such as complexity, chaos, and artificial life. Statistical methods are ubiquitous in the scientific literature. Courses in probability and statistics are required for virtually all students in the natural and social sciences. Our daily newspapers are full of statistics about such matters as lung cancer risks, divorce rates, birth control failure rates, variation in temperature, the purity of soap, etc.

Yet, despite the rapid infiltration of probability and statistics into our science and media, there is substantial documentation of the wide spread lack of understanding of the meaning of the statistics we encounter (Gould, 1991; Konold, 1991; Phillips, 1988; Piaget, 1975; Tversky & Kahneman, 1971). Even highly educated professionals who use probability and statistics in their daily work have great difficulty interpreting the statistics they produce (Kahneman & Tversky, 1982).

Besides a lack of competence and understanding, students express a great deal of dislike towards courses in probability and statistics,,an antipathy well captured by the oft-quoted line, attributed to both Mark Twain and Benjamin Disraeli: "There are three kinds of lies: lies, damn lies and statistics."

Most students first encounter the subject of probability in the form of school exercises in calculating ratios of frequencies and binomial coefficients. As a result, the subject matter of probability and statistics is seen as an assemblage of formulae to be committed to memory. When students fail to master the techniques taught to them, better methods are sought to improve their ability to calculate and apply the formulae. But very little is done in school to explore basic ideas of probability or respond to questions such as: "what is a normal distribution and what makes it useful?" or "how can something be both random and structured?" Partially because the meanings of core probabilistic notions are still being debated by philosophers of mathematics and science (e.g., Chaitin, 1987; Kolmogorov, 1950; Savage, 1954; Suppes, 1984; von Mises, 1957), it is assumed that these meanings are too hard for students to access. "Safe probability" is best practiced through formal exercises without too much attention to the meanings of underlying concepts.

There is a substantial literature concerning the topic of decision-making under uncertainty (e.g., Cohen, 1979; Edwards & von Winterfeldt, 1986; Evans, 1993; Kahneman & Tversky, 1973; 1982; Nisbett, 1980; Nisbett, Krantz, Jepson & Kunda, 1983; Tversky & Kahneman, 1974; 1980; 1984). Much of this literature documents the systematic errors and biases people display when attempting to make judgments under uncertainty. A common conclusion drawn by educators and researchers from this research is that "people just aren't built for doing probability," our intuitions are faulty and are not to be trusted. So, again, the safe practice for educators wishing their students to master the material is to instill in them a mistrust of their intuitive responses and a healthy respect for the formulae[1].

The cost of this highly formal instruction in probability and statistics is high. While the best and brightest do manage to learn to use the right statistical tests in the appropriate contexts, even they do not really understand what they are doing. They experience a kind of "epistemological anxiety" (Wilensky, 1993; in preparation, 1994) - anxiety about the nature of the knowledge they are producing and what justifies it. This anxiety leads to skepticism about the validity of statistical knowledge. Add to this mix the unscrupulous use of statistical arguments to mislead voters and consumers and we begin to understand why the subject stimulates so much distaste. The cost of this educational approach is to deprive learners from accessing core probabilistic and statistical notions which are powerful means of making sense of the world.

In this paper, I present a case study of a learner engaged in a classical probability paradox. The learner was one of seventeen interviewees studied in depth as part of the Connected Probability project. I start by briefly describing the Connected Probability project and its theoretical framework,,the Connected Mathematics research program. Part of the learning environment provided in the Connected Probability project is a computer modeling language suitable for probability investigations - a version of the language Starlogo (Resnick, 1992; Wilensky, 1993). I, then, present the probability paradox with which the subject is engaged and an account of her investigation. The paradox was selected because of its potential for engaging learners in a deeper investigation into the meaning of the concept of "random" - a fundamental concept of probability theory. I conclude by arguing three points illustrated in the case study: 1) that providing support for seriously engaging such paradoxes is an important avenue to relieving epistemological anxiety about the nature of probabilistic concepts; 2) that programming can be an effective tool for resolving mathematical paradoxes (by making their hidden assumptions[2] explicit and concrete) and 3) that through programming their own computational models (and thus making their own mathematics), learners gain a much deeper understanding of probabilistic concepts than through the use of simulations or pre-built computational models.

2. Theoretical Framework

2.1 The Connected Mathematics Research Program

The name "Connected Mathematics" comes from two seemingly disparate sources, the literature of emergent artificial intelligence (AI) and the literature of feminist critique. From emergent AI and in particular from the Society of Mind theory (Minsky, 1987; Papert, 1980), Connected Mathematics takes the idea that concepts cannot have only one meaning [3]. Only through their multiple connections do concepts gain meaning. From the feminist literature (e.g., Belenky, Clinchy, Goldberger & Tarule, 1986; Keller, 1983; Gilligan, 1977; Surrey, 1991), it takes the idea of "connected knowing": knowing that is intimate and contextual as opposed to an alienated, disconnected and formalistic knowing. In this section, I only briefly sketch the Connected Mathematics approach. A more

comprehensive description can be found in (Wilensky, 1993; forthcoming-a, 1994).

The Connected Mathematics approach is rooted in the constructionist (Papert, 1991; 1993) learning paradigm. As such, it holds that the character of mathematical knowledge, is inextricably interwoven with its genesis Ü both its historical genesis and its development in the mathematical learner. A conception of mathematics as disconnected from its development leads to the misguided pedagogy of the traditional mathematics curriculum - a "litany" of defintion-theorem-proof and its attendant concepts stipulated by formal definition[4]. In contrast to approaches that attempt to explain failures of mathematical understanding in technical or information processing terms, Connected Mathematics seeks to explain these obstacles in epistemological terms. Obstacles to understanding are failures of meaning making and since meaning is made through building connections, Connected Mathematics sees these as fundamentally failures of connection.

Paradox can be an important tool of a Connected Mathematics learning environment. The recognition of paradox, is the recognition that (at least) two conceptual structures have not been integrated. This explicit recognition is the first step in making the connections between the two structures that will resolve the paradox and, most often, thereby, generate new mathematics.

The vision of mathematics as being made and not simply received leads naturally to a role for technology. Technology is not there simply to animate received truth, it is an expressive medium Ü a medium for the making of new mathematics. It follows that we can make better use of computational technologies than simply running black-box simulations Ü we can make mathematics by constructing computational embodiments of mathematical models. The true power of the computer will be seen not in assisting the teaching of the old topics but in transforming ideas about what can be learned.

Technology here is to be construed in a broad sense Ü the notations in which we express mathematics and the mathematical concepts themselves are artifacts of the technology of the period of their creation. The emergence of new powerful computational technologies, therefore implies a radical change in both the concepts and semiotic activities of a newly contextualized mathematics.

2.1.1 Connected Mathematics and Current Standards of Mathematics Reform

Connected mathematics moves beyond reform documents such as the Standards of the National Council of Teachers of Mathematics (NCTM Standards,1991a; 1991b) in its serious reexamination of the warrant for the current mathematical curriculum (see also Confrey, 1993a). In so doing, it proposes new standards for the curriculum in terms of content, process, beliefs and context. It expands mathematics content beyond the boundaries circumscribed by school and outdated technology. New technologies are used imaginatively to make abstract mathematical concepts concrete, to explore areas of mathematics previously inaccessible and to create new mathematics (e.g., Abelson & diSessa, 1980; Abelson & Goldenberg, 1977; Cuoco & Goldenberg, 1992; Feurzeig,1989; Harel, 1992; Leron & Zazkis, 1992; Noss & Hoyles, 1991; Papert, 1972; 1980; Resnick, 1991; Wilensky, 1993).

In contrast to the NCTM Standards, which portrays an "image" of mathematics (see Brandes, 1994) as essentially a problem solving activity, the vision of Connected Mathematics is more generative,,the central activity being making new mathematics. In so doing, it fosters a culture of design and exploration,,designing new representations of mathematics and encouraging critique of those designs.

Connected Mathematics acknowledges and attends to the affective side of learning mathematics and looks critically at the role of shame in the mathematical community. Listening to learners and fostering an environment in which it becomes safe for mathematical learners to express their partial understandings[5] results in a dismantling of the culture of shame which paralyzes learners - preventing them from proposing the tentative conjectures and representations necessary to make mathematical progress. In doing so, it parts company with the literature on misconceptions which highlights the gulf between expert and novice. Instead, Connected Mathematics stresses the continuity between expert and novice understanding[6], noticing that even expert mathematicians have had to laboriously carve out small areas of well connected clarity from the generally messy terrain (see also Smith, diSessa, & Roschelle, 1994).

2.2 Connected Probability

The Connected Probability project is one major branch of the Connected Mathematics program. In the Connected Probability project, we are engaged in building Connected Mathematics environments for learning probability. As a first step towards this goal, seventeen in-depth interviews[7] about probability were conducted with learners age fourteen to sixty-four. Interviews were open ended and most often experienced by the interviewees as extended conversations. The interviewer guided these conversations so that the majority of a list of twenty-three topics was addressed. The interview topics ranged from attitudes toward situations of uncertainty, to interpretation of newspaper statistics, to the design of studies to collect desired statistics and to formal probability problems.

These topics were valuable for gaining insight into people's ideas about probability, and encouraging them to think through probabilistic issues. In one sense probability is ubiquitous in our everyday lives. Yet, since probability is fundamentally about large numbers of instances, these singular everyday experiences may not be useful for building our probabilistic intuition[8]. Rarely, in our everyday lives, do we have direct and controlled access to large numbers of experimental trials, measurements of large populations, or repeated assessments of likelihood with feedback. We do regularly assess the probability of specific events occurring. However, when the event either occurs or not, we don't know how to feed this result back into our original assessment. After all, if we assess the probability of some event occurring as, say, 30%, and the event occurs, we have not gotten much information about the adequacy of our original judgment. Only by repeated trials can we get the feedback we need to evaluate our judgments. Without the necessary feedback, it is difficult to develop our probabilistic intuitions and make probabilistic concepts concrete.

A powerful way to bridge this gap between singular experiences and probabilistic reasoning is through the use of exploratory computational environments. The processing power of the computer can give learners immediate access to large amounts of data usually distributed widely over space or time. This can provide the necessary feedback needed to develop concrete understandings of probabilistic concepts. In creating an environment for learning probability, it is therefore natural to consider computational tools.

One of the investigatory tools made available to the learners in this study was a programming language, Starlogo (Resnick, 1992; Wilensky, 1993) specially adapted for modeling probabilistic phenomena. Starlogo is a massively parallel version of the computer language Logo. It allows the user to control thousands of "turtles" on a computer screen. Each of the turtles (or agents) has its own local state and can be given its own local procedures and rules of interaction[9]. Thus, the user can model the emergent effects of the behavior of many distributed agents each following its own local rules. In particular, the key probabilistic notion of distribution can be seen to arise from the actions of many independent agents (see Wilensky, in preparation, 1994). Starlogo facilitates the design of probability experiments which allow learners to test their conjectures. They can use the feedback to modify them and clarify their underlying structures via successive refinement (Leron, 1983). The use of Starlogo to design probability experiments is in keeping with the constructionist (Papert, 1991) model of learning,,that a particularly felicitous way to build strong mental models is to produce physical or computational constructs which can be manipulated and debugged. As we shall see in the case that follows, being able to articulate a model of a probabilistic concept (through programming) can lead to rich insights into the nature of probabilistic concepts such as randomness and distribution. In contrast to consumers of ready made models, learners who construct computational models are afforded the opportunity to refine their models through debugging. Through debugging their programs they can debug their probabilistic concepts and make them concrete (Wilensky, 1991).

3. The Case Study

3.1 Overview

In this paper, I present a case study of a student engaged in exploring the meaning of randomness in the context of a particular mathematical problem. This problem, first proposed by Bertrand over a hundred years ago (Bertrand, 1889) has led a fascinating mathematical life over the last century. Over the last hundred years, mathematicians have given many different solutions (e.g. Borel, 1909; Poincar, 1912; Uspensky, 1937; Keynes, 1921; von Mises, 1964) and continue to propose new solutions and reject old arguments to this day (e.g., Marinoff, 1994). This problem, which

became known as "Bertrand's paradox"[10], engaged leading mathematicians in a debate over the range of applicability of the principle of non-sufficient reason (also called the principle of indifference - i.e., the assignment of uniform probability distributions in situations of ignorance) that was a keystone of the evolving notion of randomness. Given the attention of Connected Mathematics to the genesis of mathematical knowledge both historically and developmentally, Bertrand's paradox was a natural candidate for inclusion in this study. It was hoped that the role of Bertrand's paradox in the historical development of the notion of random would be mirrored in the development of the interviewee's thinking. Interviewees were capable of calculating more than one numerical answer to the problem. The contradiction between two or more numerical answers to a seemingly well specified probability question might then be experienced by the interviewee as a paradox. The resolution of this contradiction could then lead to a deeper and richer understanding of the notion of randomness.

In the case reported on below, the paradoxical element was introduced by the interviewer. Once that intervention occurred, no further support was needed for the interviewee to recognize it as a paradox[11]. The resolution of the paradox, however, was greatly facilitated by use of the programming language.

Some researchers and educators have recently argued that students cannot make practical use of general purpose programming languages in their subject domain learning (e.g., Soloway, 1993; Steinberger, 1994)[12]. It is my hope that the case study below will help dispel this claim and show the interesting and productive interactions that can occur between programming and learning mathematics. Indeed, it is through programming that the interviewee first "thickens" (Geertz, 1973) her understanding of "random" - by seeing the need for a process, be it computational or physical, to generate randomness. Thus, a computational idea leads to a mathematical idea, resulting in the recognition of the intimate tie between a random process and a probability distribution[13]. This thickening of the notion of random and embedding it in a web of related concepts and activities leads to a Connected Mathematics understanding of randomness[14].

3.2 The Paradoxical Question

From a given circle, choose a random chord. What's the probability that the chord is longer than a radius?

This question was, in one sense, the most formally presented question in the interview. On the surface, it most mirrored the kinds of questions that students get in school. But, because of its hidden ambiguity many rich interviews arose from it. It was particularly rich in evoking epistemological questioning and investigations into the meaning of the word random and how "randomness" connects to both mathematical distribution functions and the physical world. The question was selected because it had many possible "reasonable" answers. Among the answers (backed up by solid reasonable arguments) that interviewees gave for the requested probability were 1/2, 2/3, 3/4, SQR(3)/2. The language of the question, "choose a random chord", implies that the meaning of "random chord" is well specified,,there should only be one way to choose chords that are truly random. All of the interviewees shared this assumption. When they encountered two different seemingly correct solutions to the question that led to different values for the probabilities, they were therefore confronted with a paradox.

Each of the answers listed above is in fact the correct answer for a particular probability experiment. Depending on the physical experiment conducted, or, in the corresponding mathematical language, the initial distribution that is assigned to the chord lengths, different answers will be obtained. By exploring the paradox, learners came to see there was no unique way to specify a "random chord" (or in one interviewee's language "there's no such thing as a random chord"). Different ways of choosing chords are appropriate for different occasions. Different physical experiments lead to different probabilities and correspond to different ways of choosing chords. Depending on which method is used to select chords, different distributions of chord lengths will ensue. This leads to seeing the deep and powerful connections between randomness, distributions and physical experiments. It lays the intuitive substrate needed to create probabilistic models and to make sense of more advanced probability concepts such as probability measures.

3.3 Case Study: Ellie

Of the seventeen participants in the Connected Probability project, fifteen engaged with the random chord problem.

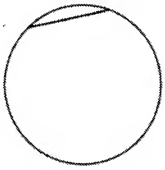
These fifteen interview fragments include many different themes and investigation paths. Each case is different. Nonetheless, the case that follows can be described as typical, if not in its specific details, then in the general outline of the investigation.

3.3.1 First Encounter

Ellie is a computer professional who has a solid undergraduate math background. Like many of the other interviewees, Ellie gets into trouble trying to understand the meaning of random. We could have resolved her difficulty by specifying a particular distribution of chords, or by describing a specific experiment to generate the chords. But had we done that, Ellie would not have developed her insights into the meaning of randomness. As teachers, it is often difficult for us to watch learners struggle with foundational questions and not "clear up the misunderstanding". However, the temptation to intervene is more easily resisted if we keep in mind that it is only by negotiating the meaning of the fundamental concepts, by following unproductive, semi-productive and multiple paths to this meaning that learners can make these concepts concrete.

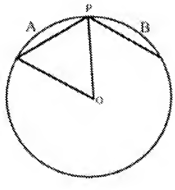
Many interviewees answered this question fairly quickly using the following argument. Chords range in size from 0 to 2r. Since we're picking chords at random, they're just as likely to be shorter than "r" as they are to be longer than "r". Hence the probability is equal to 1/2.

Ellie engaged herself with this question but approached it differently. She began thinking about the problem by drawing a circle and a chord on it which she knew had length equal to the circle's radius, as shown below. (See figure 1).



[Figure 1]

After contemplating this drawing for a while, she then drew the following figure: (See Figure 2).



[Figure 2]

With the drawing of this picture came an insight. She pointed at the triangle in the figure and said:

Ellie: It has to be equilateral because all the sides are equal to a radius. So that means six of them fit around a circle. That's right, 6 * 60 = 360 degrees. So, that means if you pick a point on a circle and label it P, then to get a chord that's smaller than a radius you have to pick the second point on either this section of the circle [labeled A in the figure above] or this one [labeled B in the figure above]. So since each of those are a sixth of the circle, you get a one third chance of getting a chord smaller than a radius and a two thirds chance of a chord larger than a radius.[15]

Ellie was quite satisfied with this answer and I believe would not have pursued the question any more if not for my prodding.

3.3.2 Introducing the Paradox

U: I have another way of looking at this problem that gives a different answer.

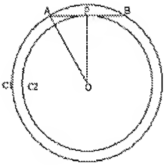
E: Really? I don't see how that could be.

U: Can I show you?

E: Sure. But I bet it's got a mistake in it and you're trying to trick me.

U: OK. Let me show you and you tell me.

I, then, drew the figure below: (See Figure 3)



[Figure 3]

U: Consider a circle, C1, of radius r. Draw a chord, AB, of length r. Then drop a perpendicular onto AB from the center of the circle, O, intersecting AB in a point, P. Then P is a mid-point of AB. Now we calculate the length of OP. We have OA = r and AP = r/2. By Pythagoras, we have OP = SQR(3)/2 * r. Now draw a circle, C2, of radius OP centered at O. If we pick any point on C2 and draw a tangent to the circle, C1, at that point, then the resultant chord has length r. If we pick a point, P', inside C2 and draw the chord which has P' as mid-point then that chord must be longer than r. Similarly, if we pick a point inside C1 but outside C2 and draw the chord which has that point as mid-point, then that chord must be shorter than r. Now pick any point, Q, inside C1. Draw a chord, EF, which has Q as mid-point. EF will be bigger than a radius if and only if Q is inside C2. It follows that the probability of choosing a chord larger than a radius is the ratio of the areas of C1 and C2. The area of C1 = pi * r^2. The area of C2 = pi * OP^2 = pi * 3/4 * r^2. So the ratio of their areas is 3/4 and therefore the probability of a chord being larger than a radius is also 3/4, not 2/3 as you said.

This explanation had a disquieting effect on Ellie. She went over it many times but was not able to find a "bug" in the argument. After repeatedly struggling to resolve the conflict, she let out her frustration:

E: I don't get it. One of these arguments must be wrong! The probability of choosing a random chord bigger than a radius is either 2/3 or 3/4. It can't be both. I'm still pretty sure that it's really 2/3 but I can't find a hole in the other argument.

U: Can both of the arguments be right?

E: No. Of course not.

U: Why not?

E: It's obvious! Call the probability of choosing a chord larger than a radius p. Then argument #1 says $p = 2/3$ and argument #2 says $p = 3/4$. If both argument #1 and #2 are correct then $2/3 = 3/4$ which is absurd.[16]

Here Ellie is quite sure that there is a definite and unique meaning to the concept "probability of choosing a random chord larger than a radius" even though she admits that she is not completely certain what that meaning is.

3.3.3 Programming

U: Would writing a computer program help to resolve this dilemma?

E: Good idea. I can program up a simulation of this experiment and compute which value for the probability is correct! I should have thought of that earlier.

Ellie then spent some time writing a Starlogo program. As she worked to code this up, she soon began to feel uneasy with her formulation. A few times she protested: "But I have to generate the chords somehow. Which of the two methods shall I use to generate them?" Nevertheless, she continued writing her program, using an approach based on argument #1. Basically, she made each turtle turn randomly and move forward a distance equal to the circle's radius to pick a point on the circle. Then, she made the turtle return to the center, turn randomly again, and move forward to pick a second point on the circle, thus defining a chord. At various points, she was unsure how to model the situation. She experimented with using the same radius for each turtle as well as giving each turtle its own radius. She experimented with calculating the statistics over all trials of each turtle as opposed to calculating it over all the trials of all the turtles. Finally, she decided both were interesting and printed out the "probability" over all trials as well as the minimum and maximum probability of any turtle.

Below are the main procedures of Ellie's program. Comments (preceded by semi-colons) have been added by the author for clarity of the exposition.

```
TURTLE PROCEDURES[17]

;;; this turtle procedure sets up the turtles[18] to setup
setxy 0 0          ;;; place myself at the origin
make "radius 10    ;;;; make my radius 10 units
make "p1x 0        ;;;; initialize temporary variables
make "p1y 0
make "p2x 0
make "p2y 0
make "chord-length 0
make "trials 0
make "big 0
make "prob 0
end

;;; This is a turtle procedure which generates a random chord.
to gen-random-chord
fd :radius                               ;;;; go to the circumference of
                                              the circle
make "p1x xpos                           ;;;; remember where I am
make "p1y ypos      bk :radius                                    ;;;; go back to the center of the
                                              circle
                                              (the origin)
```

```
rt random 360                             ;;;; turn randomly
fd :radius                                ;;;; go to a new point on the
                                               circumference of the circle
make "chord-length distance :p1x :p1y     ;;;; the chord length is the
                                               distance from where I was
                                               before to where I am now
bk :radius                                ;;;; go back to the center of
                                               the circle (the origin)
end

;;;; this turtle-procedure gets executed by each turtle at each tick of
the clock

to turtle-demon
gen-random-chord                       ;;;; choose a new chord by the
procedure above
make "trials :trials + 1               ;;;; increment the number of
chords chosen
if bigger? [make "big :big + 1]        ;;;; if the new chord is bigger than
                                            the radius,increment the number
                                            of chords chosen so far which are
                                            bigger than the radius
make "prob :big / :trials              ;;;; the probability (so far) of
                                            choosing a chord bigger than the
                                            radius is the proportion of
                                            chords chosen so far which are
                                            bigger than the radius
end

;;;; is the turtle's chord bigger than a radius? to bigger?
:chord-length > :radius                ;;;; return "true" if chord
                                            chosen is bigger than the radius
end

OBSERVER PROCEDURES

;;;; observer-demon summarizes the results of all the turtles
;;;; it gets executed at every clock tick.
to observer-demon
make "total-trials turtle-sum [:trials]      ;;;; get the total number of
                                                  chords chosen by all
                                                  the turtles
make "total-big turtle-sum [:big]            ;;;; get the total number of
                                                  chords chosen by all the
                                                  turtles bigger than a radius
make "total-prob :total-big / :total-trials  ;;;; the final probability of
                                                  choosing a chord bigger than
                                                  a radius is the ratio
                                                  of the above two totals
every 10 [type :total-big type :total-trials
          print :total-prob type turtle-min [prob]  ;;;;  print some
                                                          statistics
                                                          including the
                                                          probabilities of
                                                          the turtles with
                                                          the smallest and
                                                          largest probabilities
          print turtle-max [prob]]
end
```

Ellie ran her program and it indeed confirmed her original analysis. On 2/3 of the total trials the chord was larger than a radius. For a while she worried about the fact that her extreme turtles had probabilities quite far away from 2/3, but eventually convinced herself that this was OK and that it was the average turtle "that mattered".

But Ellie was still bothered by the way the chords were generated.

E: OK, so we got 2/3 as we should have. But what's bothering me is that if I generate the chords using the idea you had then I'll probably get 3/4[19]. Which is the **real** way to generate random chords? *(emphasis added)*

The need to explicitly program the generation of the chords precipitated an epistemological shift. The focus was no longer on determining the probability. It now moved to finding the "true" way to generate random chords. This takes Ellie immediately into an investigation of what "random" means. At this stage she is still convinced, as she was before about the probability, that there can be only one set of random chords. She assumes that the problem is to discover this unique set.

U: That's an interesting question.

E: Oh, I see. We have two methods for generating random chords-what we have to do is figure out which produces really random chords and which produces non-random chords. Only one of these would produce really random chords and that's the one that would work in the real world.

U: The real world? Do you mean you could perform a physical experiment?

E: Yes. I suppose I could. ...Say we have a circle drawn on the floor and I throw a stick on it and it lands on the circle. Then the stick makes a chord on the circle. We can throw sticks and see how many times we get a chord larger than a radius.

U: And what do you expect the answer to be in the physical experiment?

E: Egads. *(very excitedly)* We have the same problem in the real world!!! We could instead do the experiment by letting a pin drop on the circle and wherever the pin dropped we could draw a chord with the pin as midpoint. Depending on which experiment we try we will get either answer #1[20] or #2. Whoa, this is crazy. So which is a random chord? Both correspond to reality?..... This was a breakthrough moment for Ellie, but she was not done yet. Though her insight above suggests that both answers are physically realizable, Ellie was still worried on the "mathematics side" that one of the methods for generating chords might be "missing some chords" or "counting chords twice". Ellie needed to connect her insight about the physical experiment to her knowledge about randomness and distribution. She spent quite a bit of time looking over the two methods for generating chords to see if they were counting "all the chords once and only once". She determined that in her method, once she fixed a point P, there was a one-to-one correspondence between the points on the circle and the chords having P as an end-point. She concluded therefore that there "are as many chords passing through P as there are points in the circle". However, there will be more chords of a large size than chords of a small size. As could be seen from her original argument, there will be twice as many chords of length between r and 2r as there are of chords of length between 0 and r. Now, for the first time, Ellie advanced the argument that many interviewees had given first.

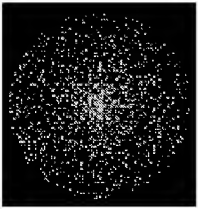
3.3.4 Reflection

E: I never thought of the obvious. I've been sort of assuming all along that every chord of a given size is equally likely. But if that were true then I could have solved this problem simply. Each chord would have an equal chance of being of length between 0 and the diameter. So half the chords would be bigger than a radius and half smaller.

Ellie went on to see that, in argument #2, large chords are more probable than small chords. She reasoned that for every chord of a given size (or more accurately a small size interval) there was a thin annulus of points that would generate chords of that size by method #2. Annuli closer to the center of the circle would correspond to large chords and annuli near the circumference would correspond to small chords. She went on to demonstrate that annuli close to the center would have larger areas than annuli close to the circumference. Thus large chords become increasingly more probable[21].

Another interesting feature: The program that Ellie wrote placed all the turtles at the origin and since Ellie, as a professional programmer, wrote state transparent code[22] they stayed at the origin. Initially, she had placed the turtles at the origin of the screen's coordinate system because she recognized a potential bug in her program. If she created the turtles in random positions as is typical in Starlogo the turtles might "wrap"[23] around the screen when drawing their circles and thus incorrectly calculate their chord lengths. But, because the turtles remained centered at the origin, the program was not very visually appealing. While we were engaged in the interview, a student came by

and watched. He asked us why nothing was happening on the screen. Ellie explained what she was investigating and then had an idea of how to make the program more interesting. She decided to spread the turtles out a bit so each could be seen tracing its circle, turning yellow if its chord was longer than a radius and green if it was shorter. To spread the turtles out without getting too close to the screen edge, Ellie told each turtle to execute the command fd random (60 - radius) telling each turtle to move a random amount out from the origin. In doing this, the result wasn't quite what Ellie had hoped for. Near the origin there was a splotch of color [mostly yellow] as all the turtles were squeezed together very tightly, while near the edges the turtles were spaced out more sparsely (as in the following figure). (See figure 4)



[Figure 4]

What had happened here quite by accident was a mirroring of the original dilemma. Ellie had used a linear random function to move points into a circular planar area. There were an equal number of turtles in each equally thick disk around the origin, but the outer disks had greater area than the inner disks and therefore appeared less crowded.

So Ellie's function which successfully spread turtles out evenly (and what she then called randomly) along a line did not spread them out evenly on the planar screen. This experience was an important component of her subsequent "aha" moment-exposing her as it did to a crack in her solid and fixed notion of "random".

4. Discussion

As can be seen from the above interview fragment, the primary obstacles to Ellie's resolving the paradox are epistemological in nature. She faced such questions as: Can a definitive probability problem admit two different numerical answers? Is the notion of a random chord well defined? What is the relationship between a physical experiment and a mathematical calculation? How do you put into correspondence an infinite number of chords and an infinite number of points? When can you say you have selected a reference set for which it is justified to say all chords in it are equally likely to be selected[24]? As Ellie's interview suggests, an important finding of the Connected Probability research is that the primary obstacles to the interviewees' facility with probability are epistemological in nature. Their difficulties stem from fundamental confusion about such notions as randomness, distribution and expectation. The epistemological status of these concepts was in doubt (What kinds of things are they? What makes them work? Are they "natural" or constructed?). As a result, many interviewees reported an inability to resolve the competing claims of conflicting probabilistic or statistical arguments. Faced with two equally compelling arguments, they are in the position of Buridan's ass: paralyzed between two equally appealing bales of hay. They can't choose either one and so never "get" any probability.

4.1 Paradox as a Learning Tool

Responses to this paralyzing situation include: 1) blaming themselves-"I just can't see why one of these is better than the other" and, in their discouragement, abandoning the domain to experts; 2) blaming the subject-"You can say anything with statistics and there's no way of proving you wrong"; 3) rejecting the importance of the conflict-"So, no big deal, they're both right"[25]. No amount of purely formal instruction in the use of probabilistic and statistical

formulae can begin to address the "epistemological anxiety" that engenders these responses. What is needed is a therapeutic intervention-the valuation of both sides of a contradictory argument together with validation of the learner's competence to resolve these competing claims. In contrast to the literature on misconceptions, it is important to emphasize the continuity between the learner's confused and messy understanding of the domain and that of the experts[26][27]. Essentially, the gist of the intervention concerning paradox is creating an environment in which learners are encouraged to pay attention to a situation in which there are conflicting probabilistic arguments, and to replace the experience of helplessness or anxiety in the face of this conflict with the feeling of excitement associated with a meaningful learning opportunity.

By the time Ellie had gotten to the circle chords question, she had already spent five and a half hours during three separate days over a three-week period in a Connected Mathematics interview. During this time, she had encountered and constructed many paradoxes and, along the way, gained confidence in her ability to resolve them to her satisfaction. She, therefore, did not need much support on this occasion in accepting the paradox as an opportunity for learning. She took both arguments (her own and the interviewer's) seriously. Even though she suspected that the interviewer's argument was a clever trick, she felt a need to find a flaw in that argument. This need to find a flaw in one side of the paradox (as opposed to just embracing the argument that seems good to her) is a powerful avenue for learning. Less sophisticated learners are content to find an argument they can believe and do not feel a need to refute any counterarguments. Seizing on the plausible argument without refuting the counterargument was a common phenomenon in the interviews and was particularly salient in discussions of the Monty Hall family of problems (Gilman, 1992; Wilensky, 1993).

4.2 Programming-Making Probability Concrete

It was not until Ellie *programmed* a simulation of the problem that she began to resolve the paradox. Note that she had already begun to see the direction of resolution before she ran her simulation, even before she completed writing the program[29]. This was a common phenomenon across interviewees. Explicitly representing the situation in which the probability problem is embedded, making it concrete, was frequently enough to resolve the present difficulty and move to the next level of subtlety[30]. Writing the code to generate the chords forced Ellie to embody the randomness of the chords in a computational process. This led her to see that different computational processes generate different sets of chords (or distributions of chord lengths). Still clinging to the idea that there was only one truly random set of chords, she moved to the level of physical simulation where surely, she thought, she could see which set of chords would really be picked out. At that point came the "aha" that there was no unique set of real and truly random chords-different physical experiments would lead to different sets of "random" chords. She had made the connections between the physical experiments, the computational processes and the mathematics of randomness and distribution. Equipped with this connected web of relations, we might venture to say that Ellie would now also be ready to deal with the formalisms of measure theory and probability measures without getting lost. The concrete foundation built up during the interview would provide support in navigating through the formalism, guiding its use and preventing its abuse.

4.3 Using Models vs. Building Models

Some researchers have argued (e.g., Soloway, 1993; Steinberger,1994) that using specialized applications, domain-specific models, and exploratory simulations can provide the benefit of programming without the "overhead" associated with learning the language. An illuminating comparison can be made between the experience of programming random chords and that of using specialized probability courseware.

One such package, ConStats (Cohen, Smith, Chechile & Cock, in press), was designed with the objective of helping students gain "a deep conceptual understanding of introductory probability and statistics" through an "active experimental style of learning" (Cohen, Chechile, Smith, Tsai & Burns, 1994). As such, it is based in constructivist principles. However, the experiments students can conduct with ConStats consist of manipulating the parameters of a preconceived model. Students cannot program in ConStats or build models to pursue questions that arise.

The software is impressive, with well implemented graphics, an easy-to-learn user interface, extensive contextual help

facilities, and a large selection of features. A principal emphasis of the software package is on distributions. The package contains many different distributions, both continuous and discrete, each with its own name and associated text describing its characteristics. In addition, for each kind of distribution, users have a host of parameters which they can manipulate and view the resultant change in the graph of the "random variable".

ConStats has both the strengths and weaknesses of the broader class of what can be called "black-box" simulations (i.e., simulations in which the user does not have explicit access to the modeling algorithm). These strengths include the ability of users to engage quickly with high level models, the availability of specialized domain specific tools, engaging user interfaces and broad coverage of the subject domain. The chief weakness is the lack of "read/write" access to the model. As a result, learners cannot explore what processes govern the way the parameters change the model. More importantly, they cannot explore the consequences of changing the structure of these processes themselves. As a result, they do not develop a solid understanding of these underlying processes.

The ConStats software was used extensively by students in a number of university-level courses. After each course was completed, the students were given a post-test designed to measure their comprehension of concepts "covered" by the software. The researchers conducting the evaluation (Cohen et al., 1994) report that conceptual comprehension was significantly greater for those students using the software than for the control group. One of the questions on the post-test was: "What is it about a variable that makes it a random variable?" The first author of the evaluation study reported (Cohen, 1993) that in all the exams he has seen, not a single student had "given the correct answer", nor had a single one mentioned the concept of distribution in his/her answer[31]. Most students just left it blank. The most frequent non-blank answer was: "a variable that has equal probability of taking on any of its possible values". Despite the fact that they had spent hours manipulating distributions and had plotted and histogrammed their "random variables," they missed the connection between these activities and the concept of random variable. The connection between distribution and randomness was perhaps too obvious to the software designers. They did not recognize the necessity of the learners constructing that connection for themselves if they are to explore it further through the software.

The ConStats software encourages exploration through changing parameters which may explain its success in improving conceptual understanding in courseware subject matter. But ConStats users' understanding of randomness is seemingly impoverished. They have not made connections between the distributions they manipulated and observed and the concept of random variable. It is unlikely that they have developed a widely-connected and intuitive sense of the concept of randomness. In contrast, most of the interviewees in the Connected Probability project developed a deeper understanding of randomness. By explicitly confronting the question of the meaning of randomness and by explicitly representing it in a program, the interviewees developed strong intuitions that were not developed by the users of the courseware.

Leaving aside the differences in conceptual understanding promoted by the two approaches, there is also an important issue of educational goals. Particularly in the area of statistics, the educational goal should emphasize interpreting and designing statistics from science and life rather than mastery of curricular materials. In order to make sense of scientific studies, it is not sufficient to be able to verify the stated model; one needs to see why those models are superior to alternative models. In order to understand a newspaper statistic, one must be able to reason about the underlying model used to create that statistic[32] and evaluate its plausibility. For these purposes, building probabilistic and statistical models is essential.

Computer-based exploratory environments for learning probability can facilitate greater conceptual understanding. The computer's capacity to repeat and vary large numbers of trials, ability to show the results of these trials in compressed time (and often in visual form), makes it possible to encapsulate events that are usually distributed over time and space. This can provide learners with the kinds of concrete experiences they need to build solid probabilistic intuitions.

A central issue, then, is between learners using pre-built models and learners making their own models. The ability to run pre-built models interactively is an improvement over static textbook based approaches. By manipulating parameters of the model, users can make useful distinctions and test out some conjectures. The results of the Connected Probability project suggest that for learners to make use of these pre-built models, they must first build

their own models and design their own investigations.

It is possible to combine the two approaches (e.g. Eisenberg, 1991; Wilensky, forthcoming-b, 1994) by providing pre-built models that are embedded in programming environments, creating so-called "extensible applications" (Eisenberg, 1991). This combined approach has the advantages of both pre-built and buildable models. The challenge of such an approach is to design the right middle level of primitives so that they are neither (a) too low-level, so that the application becomes identical to its programming language, nor (b) too high-level, so that the application turns into an exercise of running pre-conceived experiments. The metric by which the optimal level can be judged is in the usefulness to learners. This requires an extensive research program. The findings from this research must inform the development enterprise.

Introduction | Conclusion | References

5. Concluding Remarks

In the Connected Probability project, learners such as Ellie succeeded in making deep probabilistic arguments that probed at the foundations of the discipline. Having understood the foundational concepts in this deep way, they developed a strong intuitive understanding of such concepts as randomness, distribution and expectation. Solid intuitions about probability and statistics were clearly developed by learners in this study. This shows that we are not, by our natures, as some have argued, unable to reason intuitively about probability.

The Connected Probability project is an instantiation of the Connected Mathematics approach. The key elements of the Connected Mathematics approach that enabled these changes are:

- The explorations of multiple meanings of concepts and making connections between these different representations

  Like Ellie, they saw the connections between representations of randomness in different domains including physical experiments, probability distributions and computational processes.

- A focus on epistemological issues (as they specifically relate to how learners construct understandings)

  Ellie focused on what it means for something (a process) to be random? Is there only one way of choosing chords randomly or can there be multiple ways?

- The use of paradox

  The paradox reinforced the focus on epistemological issues - it placed her epistemology of mathematics in doubt. Ellie wondered: What kind of discipline is mathematics if a "unique" probability can be equal to both 2/3 and 3/4?

- Conducting a learner-owned investigation (as opposed to problem solving) as the central activity of mathematical learning

  Even though the random chord problem started out as a classic formal problem, Ellie engaged herself with it to see it as her own.

- Acknowledgment of and attention to the affective side of learning mathematics

  Ellie would not have engaged herself with the paradox had she not been encouraged to believe in the pursuit enough to overcome the epistemological anxiety that usually prevents learners from getting so engaged. Crucial to this self-confidence is a "cognitive-emotive" therapy for the sense of shame produced by a mathematical culture that prevents learners from expressing the epistemological anxiety and tentative understandings that are

at its root.

- Making mathematics (and articulating it in a concrete form)

  Ellie is encouraged to create alternate representations of the problem and work out definitions of randomness that make sense to her. This enables her to see mathematics as a personal odyssey of meaning making, not an externally given corpus to be assimilated but not affected by her.

- The use of technology as a medium for making and articulating mathematics

  Ellie was able to design her own experiment to explore the different sides of the paradox. This ability to express her partial understandings of "random chord" in a computational model was key to the refinement of her mental model and provided a powerful semiotic context for her articulation of her mathematical thought.

The availability of the programming environment facilitated many of these goals. The programming environment facilitated Ellie's conducting her own investigation. It provided a language, a different notation in which Ellie could express her mathematical ideas. It provided a signing environment, a place-holder for these ideas to exist outside of Ellie. And because this language is dynamic- it can be "run"- it provided feedback to Ellie's ideas. This trialogue between Ellie's mental model, the expression of her mental model in encapsulated code and the running of that code, allowed Ellie to successively refine the creative structure of her thought. While one might concede that it is theoretically possible for Ellie to have resolved her problem with a pre-built model in which a randomizing parameter was modified, the leaner-modeling approach is clearly significant in its outcome and arguably more practical in its implementation. This is because, for Ellie to have come to a similar set of insights, a model designer would have had to anticipate all of Ellie's concerns and built them into the model. Clearly, this is impossible to do in general - educational software designers cannot anticipate all the directions that a learner might want to investigate and incorporate them into a parameter model. Moreover, users of "parameter-twiddling" software realize that they are pursuing someone else's investigation. This realization decreases the motivation of discovery. Lastly, such closed environments reinforce a view of mathematics learning as a process of verifying already known mathematics as opposed to seeing it as a personal odyssey of mathematics making. In designing computer-based environments for learning probability, we must remember that allowing users to create their own models is necessary for truly learner-owned investigations.

For many learners in the Connected Probability project, this experience of doing Connected Mathematics was so different from their experience in regular mathematics classrooms, that they did not recognize their activity as being mathematics. Learners who had "always hated mathematics" and had been told that they were not "good at mathematics" were excitedly engaged in doing mathematics that could be easily recognized by mathematicians as "good mathematics". Having created a strong intuitive foundation for the conceptual domain, learners could also go on to engage the formal approaches and techniques with an appreciation for how they connect to core idea of probability and statistics. Even more importantly, they now understood that mathematics is a living growing entity which they could literally make their own.

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Introduction | Conclusion | References

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[1] In a graduate probability course at MIT, the professor explicitly admonished the class members not to try to do inverse probabilities in their heads since their intuitions were not reliable. Instead, he said, always use the Bayes formula to calculate inverse probabilities.

[2] Assumptions implicit in the formulation of the paradox or in the preconceptions of the learner.

[3] A nice example of the many meanings of "derivative" can be found in a recent paper by Thurston (1994).

[4] Formal proof and definition is an after-the-fact reconstruction of the processes of coming to know in mathematics. The justification of such reconstruction for the purposes of communication within expert culture is certainly allowed. What is unfortunate and damaging pedagogoically is that this re-presentation becomes an active conception of what mathematics is and what it is to know mathematics.

[5] The taboo against expressing partial understandings is endemic to school discourse. To break it, teachers must explictly model expressing their own confusions and groping for clarity. One reason this is hard to do is that it is very difficult to remember what it was like not to grasp a mathematical concept that is now self evident. There are many striking parallels between the development of conservation in children (Piaget, 1952) and the acquistion of new mathematical concepts. One feature they share is the inconceivability of one's previous understanding-what is it like to think that there is more water in a tall glass than there was in the shorter glass which you emptied into the taller container? [for further discussion, see Wilensky, 1993].

[6] Because it suggests that making is endemic to mathematical activity, the Connected Mathematics view is that: learners make connections, they don't cross intellectual ravines. Thus the process of becoming expert in mathematics is one of adding connections and not removing or replacing novice knowledge.

[7] The shortest interview was two hours long, the longest eighteen hours and the median seven hours. These figures refer to the face-to-face interview time. Some interviews continued over electronic mail for up to two months following face-to-face interactions.

[8] Part of what makes an event singular is that we do not interpret it as a member of a class of events. It is only when we can stand at a distance from the event and see it in the context of many other events, that we can begin to make the reference classes needed to make probabilistic judgments.

[9] These "object-oriented" features of the language make StarLogo a more accessible environment for modeling. In contrast to other modeling environments, such as STELLA (Richmond & Peterson, 1990), which model with aggregate quantities and flows, StarLogo is "object" based - thus facilitating concrete interactions with the basic units of the model.

[10] The name "Bertrand's paradox" was given by Poincare.

[11] This was true in roughly half of the interviews in this study. The later in the interview the paradox occurred, the

more likely that it was recognized and owned.

[12] A weaker form of this claim is that programming requires too much "overhead" that distracts learners from the mathematics at hand. This paper does not respond directly to this weaker claim. Let me note briefly that: 1) Logo and Starlogo are conceived here as lifelong tools and powerful expressive media across many domains, not just probability. 2) In contrast to languages such as Fortran or Basic, meaningful Starlogo programs are usually quite short and StarLogo has "low threshold" (i.e., easy for novices to write meaningful programs) as a primary language design criterion (Papert, 1980; Resnick, 1991).

[13] Or as some interviewees saw it, each process leads to a different "meaning" of random.

[14] A positivist or strict formalist critic might object that in fact the notion of randomness has been replaced by a more precise and technical notion. In practice, however, the new ideas coexist with the old and take much of their sustenance from their connections to prior conceptions and other contexts for recognizing randomness.

[15] The transcripts have been "cleaned up" some (removing pauses, umms and many interjections) for clarity of the exposition. Bracketed comments are the author's clarifying remarks.

[16] At this point, Ellie actually wrote down a formal mathematical proof by contradiction. The last line of the proof was: Therefore 2/3 = 3/4. Contradiction.

[17] The Starlogo procedures are divided into turtle procedures and observer procedures. Turtle procedures are executed by each turtle in parallel. Observer procedures set up the general environment and summarize the behavior of turtles.

[18] In this case, each turtle sets itself up at the origin on the circumference of a circle of radius 10.

[19] Ellie did go on and write the code to do this experiment just as a check of her insight. Her new code is the same as the old code except for a rewrite of the procedure "gen-random-chord".

[20] I chose not to intervene at this juncture and point out that the first experiment Ellie proposed did not correspond exactly to her first analysis and method of generating chords.

[21]

EXHIBIT D

What is Normal Anyway? Therapy for Epistemological Anxiety

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Introduction | Conclusion | References

1.0 Introduction

Concepts and intuitions from probability and statistics pervade many aspects of our "non-mathematical" lives. When we decide whether to pay more for a car that is reported to be "more reliable"; when we choose to safeguard our children by living in a "low-crime" neighborhood; when we change our diet and exercise habits to lower the risk of heart disease; when we accept or refuse a job with other options pending; all the above actions involve implicit probability judgments. Arguably, no other area of mathematics can be said to apply so widely. No other area is so potentially empowering- so useful in making decisions, both intimate and professional, and affords us such opportunities for making sense of phenomena, both natural and social.

Yet, as the research literature (discussed below) has made abundantly clear, students do not take great advantage of these powerful ways of making sense of the world. This paper seeks to give an account of this missed opportunity in terms of the framework of Connected Mathematics (Wilensky, 1993).The paper begins with a sketch of background issues related to learning probability and statistics. The Connected Probability Project (Wilensky, 1993; 1995a; 1995b) is then described and situated in the framework of Connected Mathematics. The notion of "epistemological anxiety" as a primary obstacle to learning probability and statistics is articulated and developed. The paper then describes how the use of computational environments -- specifically object-based parallel modeling languages -- can allow learners to successfully address epistemological anxiety in the realm of probability and statistics. The remainder of the paper seeks to instantiate the theory and practice of Connected Mathematics in case studies concerning learning about the concept of normal distribution. These cases illustrate how traditional teaching of the concept of normal distribution which relies on formalism and macro- level summary statistics leads to epistemological anxiety. The cases also illustrate how Connected Mathematics engages issues of epistemological anxiety and how it fosters a deeper understanding of normal distributions. This understanding is advanced through learners modeling normal distributions as emergent phenomena.

1.1 The Subject Learners are Required to Hate

People's attitudes towards probability and statistics can be summed up in the well worn adage attributed to both Twain and Disraeli: "There are three kinds of lies: lies, damn lies, and statistics." (in Tripp, 1970)

Probability & Statistics (henceforward, P&S) research "methods" courses have achieved notoriety in the undergraduate and graduate curriculum as the bane of many a student. Students in the social sciences are usually required to take a statistics course. Though such courses are intended to introduce them to concepts and methods which they can then employ in making sense of real data, they frequently report being able to work the textbook problems but having little idea of "what they mean," or how to apply them to novel situations (see, e.g., Phillips, 1988; {Rubin et al, 198x}). Students complain about being able to make several different arguments to solve a problem, all of which sound plausible yet yield different answers (Wilensky, 1993). How are they to choose among the plausible alternatives?

This confusion and indecision leads to what I have termed "epistemological anxiety" (Wilensky, 1993)- a feeling, often in the background, that one does not comprehend the meanings, purposes, source or legitimacy of the mathematical objects one is manipulating and using.

1.2 Responding to a literature of Innate Constraints

There has been considerable research (e.g., Gould, 1991; Konold, 1991; Phillips, 1988; Piaget, 1975; Tversky & Kahneman, 1974) documenting difficulties people have in learning, understanding and using concepts of probability and statistics. Much of this research locates the source of the problem in cognitive constraints of the mind of the learner (see e.g., Cohen, 1979; Edwards & von Winterfeldt, 1986; Evans, 1993; Gould, 1991; Kahneman & Tversky, 1973; 1982; Nisbett, 1980; Nisbett et al, 1983; Tversky & Kahneman, 1974; 1980; 1983).

In their now classic work, Tversky & Kahneman (1982) document the persistent errors and "misconceptions" that people make when making probabilistic "judgments under uncertainty". Among these errors are systematic misconceptions about probabilities of conjunctions, inverse probabilities, updating probabilities in the face of new evidence and "seeing" non-random pattern in random data. These systematic errors are repeatable and don't seem to go away even when people have had significant training in probability. This contributes to a widespread belief that humans are incapable of thinking intuitively about probability.

Tversky & Kahneman speculate as to the origin of these systematic biases in people's assessment of likelihoods. They theorize that people's mental resources are too limited to be able to generate probabilistically accurate judgments. Consequently, people are forced to fall back on computationally simpler "heuristics". In other words, we have been "hard-wired" not be able to think about probability and must circumvent our natural thinking processes in order to overcome this liability.

The view that humans must circumvent natural thinking processes in the realm of probability has spawned a large literature and has become very influential. For the purposes of this exposition, let us call this view the "accommodationist" view. The effect of this accommodationist view on P&S instruction has been a deference to the mechanical manipulation of the tokens of formal notation as a guard against unreliable intuitions. This lesson for P&S instruction has reached the highest levels of education. In an introductory graduate course in probability and statistics at a major research university, the professor wrote down Bayes theorem for calculating inverse probabilities and then baldly announced:

> *"Don't even try to do inverse probabilities in your head. Always use Bayes formula. As Tversky and Kahneman have shown, it is impossible for humans to get an intuitive feel for inverse probabilities".* (in Wilensky, 1993).

This quote is a pedagogical embodiment of the accomodationist view.

It is possible to begin with a more conservative set of assumptions then do the accommodationists. We could assume that P&S is an area of mathematics not widely separated from the rest of the discipline. Furthermore, we could posit a broadly developmental view of mathematical intuition which allows for conceptual change. We would then look elsewhere for the source of learners' difficulties with P&S. We would look at the learning methods, tools and cognitive technologies (Pea, 1987) available to the learner. Given the prevalence of this accomodationist view, it would be of great interest if it could be shown that a suitable learning environment could help learners transform their probabilistic biases into effective intuitions and a solid conceptual understanding of P&S. This work is being undertaken through the Connected Probability project.

2.0 Connected Probability

The mission of the Connected Probability project (Wilensky, 1993; 1995-a; 1995-b) is to better understand the source of learners' difficulties in P&S and to build learning environments that would foster the development of intuitive conceptions of foundational concepts and positive attitude towards the discipline. The Connected Probability project was launched with the hypothesis that learners' difficulties with P&S are not primarily due to hard-wired cognitive constraints. We conjecture that students' antipathy towards P&S is primarily due to epistemological anxiety concerning the basic concepts of the discipline. We further locate the source of the anxiety in the lack of sufficiently powerful learning environments for P&S. The paucity of the learning environment stems, in great part, from the unavailability of powerful tools for experimentation and construction in probability. The anxiety is further reinforced by the teaching practices employed in mathematics classrooms and a "protective" culture which discourages revelation of mathematical conceptions and process and its consequences for mathematical discourse in the classroom. Given this "diagnosis", it follows that learners' difficulties in P&S can be effectively addressed without resorting to either rigidly formalized instruction or neural surgery.

In the initial year of the project, seventeen in depth interviews about P&S were conducted with learners age fourteen to sixty-four. Interviews were open ended and most often experienced by the interviewees as extended conversations. The interviewer guided these conversations so that the majority of a list of 23 topics was addressed. The interview topics ranged from attitudes toward situations of uncertainty, to interpretation of newspaper statistics, to the design of studies for collecting desired statistics and to formal probability problems. In most interviews, a computational modeling environment designed for experimenting with P&S was made available to the learners.

2.1 Probability Distributions

The interview format was designed to focus on several key concepts in P&S. The notion of Probability distribution is one such key concept. We focus on probability distribution, partly because of its importance: 1) an understanding of probability distribution is crucial to understanding the statistical models ubiquitous in scientific research. 2) The concept is equally central to participation in the public forum as an informed citizen. 3) Without the concept of distribution learners cannot truly understand how events can be both unpredictable and constrained - we cannot have a coherent concept of randomness (Wilensky, 1993; 1995-b). 4) Probability distributions stand at the interface between the traditional study of probability and the traditional study of statistics and, thus, afford an opportunity to make strong connections between the two disciplines. Another reason we chose to focus on probability distribution is the potential for bringing about meaningful change in the learning experience for many students. In a typical course in probability and statistics, students are exposed to a standard library of distributions and associated formulae, but do not have a chance to construct these distributions and understand what "problems they are trying to solve". The availability of new computational "object-based parallel modeling languages" (Wilensky, forthcoming) affords learners the ability to construct these distributions as patterns emergent from probabilistic rules. Through these constructions, learners can make connections between probabilistic descriptions of discrete phenomena and the statistical descriptions of the ensemble -- in the process seeing the utility of both

descriptions. Meaningful intervention is possible.

An analysis of the interview data revealed persistent confusions on the part of most interviewees about probability distributions. These confusions and attendant anxieties resulted in the interviewees turning off their sense-making with regard to probability distributions and merely invoking rote formulae when applying probability distributions to real world problems.

3.0 Connected Mathematics

Central themes from the Connected Mathematics research program.(Wilensky, 1993; 1995c) inform the Connected Probability project. The Connected Mathematics program takes broad aim at both our view of the discipline of mathematics and the practice of mathematics pedagogy. Connected Mathematics is situated in the Constructionist learning paradigm (Papert, 1991; 1993). The name Connected Mathematics comes from two seemingly disparate sources, the literature of emergent Artificial Intelligence (AI) and the literature of feminist critique. From emergent AI and, in particular, from the Society of Mind theory (Minsky, 1987; Papert, 1980), Connected Mathematics takes the idea that a concept cannot be intelligible if it has only one meaning, it is through connections that concepts gain meaning. The feminist literature (e.g., Belenky et al., 1986; Keller, 1983; Gilligan, 1977; Surrey, 1991), contributes the idea of "connected knowing", a personal form of knowing that is intimate and contextualized as opposed to an alienated and disconnected formal knowing. Mathematical concepts derive their meaning and their power through their embeddedness in a personally and socially constructed web of connections to other ideas and experiences, both mathematical and non-mathematical.

In a Connected Mathematics learning environment, the focus is on learner-owned investigative activities followed by reflection. Thus, students are not led through the mathematical "litany" of definition, theorem proof. Mathematical concepts are not simply given by formal definitions. Instead, mathematical concepts are multiply represented (e.g., Kaput, 1987 ; Von Glaserfeld, 1989) and the environment supports multiple styles and ways of knowing (see e.g., Turkle & Papert, 1991). Mathematical intuitions are not assumed to be static, nor are some mathematical concepts assumed to be "abstract" and thus not amenable to intuitive apprehension. Learners are supported in building and developing their mathematical intuitions over a lifetime and, through this construction process, mathematical objects are seen to be more concrete (Wilensky, 1991; 1993) as learning progresses.

Connected Mathematics views mathematics as something human make ã we build tools to operate upon and make sense of our world . In contrast to learning procedures or formalisms first, as in the traditional curriculum, Connected Mathematics calls for making many more connections between mathematics and the world at large as well as between different mathematical domains throughout the learning experience(e.g. Cuoco & Goldenberg, 1995). Empowering technology is central to this effort. Technology is used as a personally expressive medium - to explore areas of mathematics previously inaccessible, to make abstract mathematical concepts concrete, and to create new mathematics In contrast to reform documents such as the NCTM standards, which portrays an "image" of mathematics as essentially a problem solving activity, its vision of mathematics is a more generative one - the central activity being making new mathematics. A culture of design and critique is developed.

Connected Mathematics pays great attention to the affective side of learning and doing mathematics. As such, it pays attention to the role of play, joy wonder and curiosity in the learning of mathematics and to the role of desire for social connection as a motivation for engaging in mathematical activity (see also Thurston, 1992). Difficulties in learning mathematics are also examined from an affective perspective. Connected Mathematics looks critically at the role of shame in mathematical culture -- how shame creates, first, anxiety (see e.g., Chipman, Krantz & Silver, 1994; Steele, 1995; Tobias, 1993) and, ultimately, negative mathematical self-image (Dweck & Leggett, 1988; Steele, 1995) in the individual learner and how it stifles

discourse and enforces hierarchy (O'Connor, 1993) in the mathematical community. In contrast, a Connected Mathematics learning environment fosters an atmosphere in which it is safe for mathematical learners to express their partial understandings and values these understandings regardless of their degree of correspondence with the mathematics canon. In doing so, it parts company with the literature on misconceptions which highlights the gulf between expert and novice. Connected Mathematics, instead, stresses the continuity between expert and novice understanding. This continuity obtains both in their explicit conceptions - novice conceptions are not discarded on the road to expertise but rather repurposed, and in their general "messy" character -- even for expert mathematicians, only small areas of clarity have been laboriously carved out from the generally messy terrain. (see also Smith, diSessa & Roschelle, 1994).

4.0 Sources of Epistemological Anxiety and the Making of Mathematical Culture

As discussed above, learners of probability suffer from an attendant anxiety about the source of legitimacy of the probabilistic procedures and formulae they are learning. This anxiety is not unique to P&S. Indeed, it is quite common in mathematics instruction.

The sources of this anxiety are multiple. The culture of the mathematics classroom contributes greatly (Hoyles, 1985; Lampert, 1991; Noss, 1988; O'Connor, 1993). Its insistence on unitary definitions for mathematical concepts reinforces the belief that mathematical knowledge is completely disconnected from the rest of the learner's knowledge (Cuoco, Goldenberg & Mark 1995; Minsky, 1987-b; Wilensky, 1993). The learner , thus poised at the lip of a yawning chasm separating her previous conceptions from the new knowledge, is understandably anxious. The formal definitions sever the links to the many related conceptions that could serve as bridging mechanisms and could alleviate anxiety. The anxiety is further enhanced by social isolation. The insistence on answers or results, absent of intellectual texture, and invalidation of personal voice (Confrey, in press; Gilligan, 1977) discourages the learner from expressing her partial and incomplete understanding. This, in turn, can create the sense that everyone else is clear about the meaning of these objects and that you alone are confused. The further knowledge that any admission of confusion will be used to rank you in the mathematics hierarchy and judge your mathematical intelligence makes it practically taboo to share your "messy" concepts (Papert, 1972; 1993; Wilensky, 1993) and attendant epistemological anxiety.

Connected Mathematics provides several sources of therapy for epistemological anxiety. Instead of formal definitions, it emphasizes connections to the learner's knowledge that make the transition to new knowledge both safer and more meaningful. By exposing the universality of confusion and messy concepts, it reassures the learner that her predicament is "normal" and shared. This, in turn, leads to the voicing of previously unvoiced concepts and sets the context for the social negotiation of mathematical knowing. This emphasis on making, voicing and sharing takes public the activity of understanding mathematics. These changes in mathematical culture are both supported and made much more realizable by the advent of computational modeling environments.

4.1 Epistemological Anxiety in the Realm of Probability

There are several reasons why epistemological anxiety is particularly pronounced in the domain of P&S:

1) We are living in a time when the meanings of the basic notions of probability theory, the ideas of "randomness", "distribution", and even "probability" itself are still quite controversial. There is, as yet, no consensual agreement on the foundational concepts of probability † they are still being debated by mathematicians and philosophers (see. e.g., Cartwright, 1987; Gigerenzer, 1987; Savage, 1954; Suppes, 1984; Von Mises, 1957). There is particular disagreement on the applicability of probability to individual instances. This leads to confusion about the applicability of probability theory to the unique situations in our lives. It raises doubts about its utility in making individual choices which might serve to animate our interest

in the subject.

2) When we encounter statistical data (as we commonly do in newspaper articles), it is often detached from its method of collection. If a statistic is left vague and disconnected in this way, we can not operate on it, transform it and compare it with others so as to make it meaningful and concrete.

3) Short term memory limitations make it difficult to collect large amounts of data in "real time". So instead, we construct summaries of the data. These summaries are not analyzable into their constituent elements and thus necessarily remain disconnected from the original data. In this way P&S is at a stage of its development not unlike other areas of mathematics throughout history. These same kinds of memory constraints once made arithmetic very difficult until memory saving technologies (such as zero, place value and generalized decimal notation) were invented (See e.g., Kline, 1972; McLuhan, 1964; Rotman, 1987). A central conjecture of the Connected Probability project is that computational technology should do for probability what Hindu-Arabic notation has done for arithmetic. Computational environments allow large bodies of data to be visualized at once in a small space and large numbers of repetitions to happen in a short time. As a result, short term memory resource limitations can be overcome. The focus can then be on probabilistic reasoning.

4) It is very hard to get feedback as to the adequacy of our probability judgments. If we assess that the probability is 30% that it will rain tomorrow and it does rain, what have we learned? We require many, many such observations in order to calibrate our judgments to the data. Again, memory limitations block us from receiving the necessary feedback, Moreover, there is little opportunity for control because the world is constantly changing and we cannot repeat the experiments so as to get systematic and controlled feedback 5) Seeing our lives as experiments in probability requires seeing our current situation as one of a large collection of similar possible situations. To make this large collection into a "concrete" object of thought (Wilensky, 1991) requires a massive construction job. It requires forging links between our current situations and situations forgotten in the past, situations not yet arisen, and counterfactual situations, (those situations which could have arisen but did not) .

6) A powerful way to think about probability situations is to think of multiple, interacting, distributed entities. Again we cannot keep so many entities in working memory (Case, 1993; Miller, 1956) at once. Consequently, we construct wholes out of the many interacting parts and understand the behavior of the ensemble in terms of its average behavior.

Memory and resource limitations, absent a notation or medium, do contribute to why, until now, many people have not developed robust probabilistic intuitions. We have reached a time where these limitations will no longer hold sway in P&S. The emergence of computational modeling environments promises to provide a holding environment , a symbolizing medium which can express data distributed over space and time. This should enable us to see view dynamic properties of ensembles and to conduct experiments with immediate feedback. It is for this reason that one of the most powerful forms of therapy for epistemological anxiety is access to a computational environment in which the objects whose epistemological status are in doubt can be modeled, experimented with and debugged to the satisfaction of the learner. The computer serves as a setting and symbolizing medium for building mathematical intuitions.

5.0 Computational Modeling in Mathematics and Science Education

In a computational modeling approach to mathematics and science education, a modeling language and sets of associated tools are made available to learners. Learners then choose a concept or phenomenon and create a computational model of the phenomenon. In contrast to pre-built simulations, where the learner is interacting with an expert model, the model-building approach allows the learner to own and pursue personally meaningful investigations. In a model building approach, there are no "black boxes" in the phenomenon of interest - the learner constructs her own "boxes".

The modeling approach has both cognitive and affective benefits. By building computational models of everyday and scientific phenomena, learners can develop robust mental models of the underlying probability and statistics. The feedback provided by building and then testing the computational model supports the learner in debugging and successively refining (Leron, 1994) the mental model. On the affective side, the modeling environment supports the learner's expression of partial understanding. In a modeling environment, it is clear that partial understandings are the norm -- one builds models by, initially, expressing a tentative model in the modeling language, then gradually refining and debugging the model. In this way modeling activities short circuit the "whole cloth" answer orientation of the culture of classroom mathematics (Sfard & Leron, in preparation). Indeed, the activity of model building is so different from normal mathematics for most students that they, often, do not identify themselves to be "doing mathematics". ({Feurzeig, 19xx; Mandinach & Cline, 1994; Thornton, 19xx}). Embedded in a suitable social environment, the opportunity to express partial understandings without judgment can go a long way towards alleviating a sense of mathematical shame. Relieved from the necessity of "getting it right" the first time, even learners not usually considered good at mathematics and science can build models that demonstrate a qualitatively greater level of mathematical achievement than is usually found in mathematics classrooms. Computational modeling serves as a powerful therapy for epistemological anxiety.

5.1 Modeling Emergent Phenomena

One productive domain for computer modeling is "emergent phenomena", in which global patterns emerge from local interactions. Emergent phenomena can provide rich contexts for learners to build with probabilistic parts. Learners can explore the stable structures that emerge when probabilistic behavior is given to distributed computational agents. Thus, instead of encountering probability through "solving" sets of decontextualized combinatoric problems, learners can participate in constructionist activities - they design and build with probability.

Probability distributions can be seen as canonical cases of emergent phenomena. They are stable global structures that arise from the interactions of multiple distributed agents. Typically, in statistics classes, the emergent nature of distributions such as the normal distribution (or "bell curve") is hidden. We "learn" about distributions through descriptions of their global characteristics (e.g., mean, standard deviation, variance, skew, moments). This conceals the way these distributions are built up or emerge from individual instances. As a result, the critical connection between the micro- level of the phenomenon and the macro- level is severed. The learner is given the macro- description and formulae and is asked to accept on authority that these macro- descriptions are appropriate for such and such a set of phenomena. This supplanting of experimentation with external authority contributes significantly to epistemological anxiety.

5.2 Object-based Parallel Modeling

The modeling environment used in the Connected Probability project, is the language StarLogo (Resnick, 1992; 1994, Wilensky, 1993; 1995b) extended to be especially useful for modeling phenomena in P&S. StarLogo is an extension of the Computer language Logo. In Logo, a graphical turtle is controlled by issuing movement commands. In StarLogo, however, the user can control thousands of "turtles" or "agents". StarLogo is an example of a new kind of modeling language ã it is an object-based parallel modeling language. Other examples of such languages include KidSim (Smith, Cypher & Spohrer, 1994) and AgentSheets (Repenning, 1993). "Parallel" means that the agents behave as if they are all executing their commands simultaneously. StarLogo was originally implemented on the Connection Machine, a massively parallel supercomputer, and was thus in fact parallel, each agent controlled by a separate processor. In more recent implementations, StarLogo runs on serial machines such as the Macintosh computer, so it is running as a parallel "virtual machine" on top of a serial architecture.

"Object-based" means that each agent is self-contained: it has its own internal state and communicates with

other agents primarily by local channels - agents don't do much action at a distance. The computer language Logo had a single such object - the "turtle". Papert (1980) argued that the power of the turtle to facilitate learning geometry lay in the fact that the child could identify with the turtle - enabling "syntonic" learning. Object-based parallel modeling languages such as StarLogo afford greater identification with their objects, and thus, in contrast to more procedural languages, foster syntonic learning of emergent phenomena.

Learners can assign micro- level rules to simultaneously control thousands of such turtles. Macro-phenomena emerge in relation to these micro- rules. This makes both forwards and backwards modeling possible - that is 1) exploring the global effects of various sets of local rules. and 2) trying to produce a known global phenomenon through finding appropriate local rules (Wilensky, 1995a).

Such object-based parallel modeling languages can be powerful environments for doing P&S. One way to think of these agents is as multiple sided irregular dice. At each step in running a model built in one of these languages, thousands of dice are thrown and results visually displayed. This use of object based parallel modeling languages is reminiscent of the technique of Monte Carlo simulations and shares feature with resampling statistics approaches (Diaconis & Bradley 1983; Konold, 1994; Simon et al, 1976; Simon & Bruce, 1991). Unlike these other approaches, in object-based parallel modeling languages, control of the program is distributed through the agents rather than centrally controlled and the agents themselves are inspectable, visualizable, "concrete" objects.

Even at this basic level of "dice rolling", many experiments in probability and statistics are suggested. Indeed, most of the standard elementary curriculum in P&S can be easily modeled in StarLogo using these "dice-rolling" features of the language alone. In a non-computational setting, experiments with large data sets are perforce constrained to be summarized by mediating global formulae and statistics. Experiments can be done for small data sets and small numbers of trials and is typically confined to several con-flips, dice-rolls and spinner-spins. But once the scale is increased, the ability to experiment is gone. As discussed earlier, the critical connection between the micro- level of the phenomenon and the macro- level is severed.

Additionally, in object-based parallel modeling languages, the "dice" can also interact with each other. As a result, many complex interactions are possible - the results of which are not predictable using standard mathematical models. The ability to experiment with such complex interactions and get visual feedback however, allows learners to make qualitative sense of different possible patterns, classifying meta-patterns and noting trends such as feedback cycles, critical densities, clustering, etc. Even though these agents -- thought of as dice -- are probabilistic components whose individual states cannot be predicted, the emergent properties of the collection of agents can be stable and predictable. This insight is essential to grasping the key notion of probability distribution. For this reason, object-based parallel modeling languages are excellent constructionist environments for learning probability -- the learner is constructing objects using probabilistic components. This situation is analogous to the acquisition of print literacy as it includes both reading and writing. Instead of just "reading" about probability (by being given the summary formulae), the learner "writes" with probability.

6.0 Case Studies

The remainder of this paper will be devoted to two case studies. The cases, taken from Connected Probability interviews, are offered as illustrations of epistemological anxiety concerning the notion of normal distribution and, in one case, a successful therapy through StarLogo modeling in a Connected Mathematics context.

6.1 Normal Anxiety - Lynn's tricky distributions

Let me introduce Lynn, a psychologist in her mid-thirties who at the time of her interview had recently received her Ph.D. For her dissertation, Lynn had done a statistical comparison study of different treatment

modalities for improving the reading comprehension of dyslexic children. While in graduate school, she had taken an introductory probability and statistics class as well as two classes in statistical methods. As such she would "naturally" be classified as having a fairly sophisticated background. In this interview fragment we will see that despite her sophisticated background, basic ideas of randomness and distribution are alienated for her - neither trusted nor appropriated for her ends. Even though Lynn was capable of applying formal statistical methods in her coursework, she was fundamentally confused about the "madness behind the method". In this interview, she starts to negotiate meaning for "normal distribution". While her attempts take her to a position vis a vis distributions which would be considered just plain wrong in most university courses and may indeed be incoherent, she is for the first time grappling with the meaning of the fundamental ideas of randomness and distribution. In so doing, she takes a step towards developing intuitions about and appropriating these concepts.

As background to the interview, I asked Lynn about her attitudes towards mathematics. Her reply was indicative of the issues that would arise in the body of the interview:

U: So, what was math like for you in school?

L: Well, I was always good at math. But, I didn't really like it.

U: Why was that?

L: Why? I donêt know. I guess I always felt like I was getting away with something, you know, like I was cheating. I could do the problems and I did well on the tests, but I didnêt really know what was going on.

One of the first questions that arose in the body of the interview with Lynn was: What would you estimate is the probability that a woman in the US would be at least 5'5" tall? Here is the text of the ensuing conversation:

L: Well I guess it would be about 1/2.

U: Why do you say that?

L: Well height is normally distributed and I'd say the mean height of women is about 5'5" so half of the women would be taller than 5'5".

U: Why would half the women be taller than the mean?

L: Because the curve of the normal distribution is symmetric around the mean - so half would be below it and half above it.

U: What about the number that are exactly 5'5" tall?

L: Well I guess they could make a difference, but no - they shouldn't matter because they're just one data point and so can be ignored.

U: You can ignore any one data point?

L: Yes, ... because there are infinitely many data points so one doesn't make a difference.

U: But can you make infinitely many height discriminations? How do you measure height?

Here, I'm just trying to probe Lynn's thinking about discrete vs. continuous distributions.

L: Well.... I guess we measure it in inches -- so there probably aren't infinitely many data points. I'm

somewhat confused because I know height is distributed normally and I know that for normal distributions the probability is 0.5 of being bigger than the mean, but how come you can ignore the bump in the middle? I guess 0.5 is just an estimate, it's approximately 0.5.

U: How do you know height is distributed normally?

L: I don't remember where I first learned it, but lots of the problems in the stats books tell you that.

My question was intended to elicit an explanation of why Lynn believed height was distributed normally in terms of her personal knowledge, but Lynn responded to it as about probing the context of gaining that knowledge and the authority it derives from.

U: Why do you think height is distributed normally?

L: Come again? (sarcastic)

U: Why is it that women's height can be graphed using a normal curve?

L: That's a strange question.

U: Strange?

L: No one's ever asked me that before..... (thinking to herself for a while) I guess there are 2 possible theories: Either it's just a fact about the world, some guy collected a lot of height data and noticed that it fell into a normal shape.....

U: Or?

L: Or maybe it's just a mathematical trick.

U: A trick? How could it be a trick?

L: Well... Maybe some mathematician somewhere just concocted this crazy function, you know, and decided to say that height fit it.

U: You mean...

L: You know the height data could probably be graphed with lots of different functions and the normal curve was just applied to it by this one guy and now everybody has to use his function.

U: So youêre saying that in the one case, it's a fact about the world that height is distributed in a certain way, and in the other case, it's a fact about our descriptions but not about height?

L: Yeah.

U: Well, if you had to commit to one of these theories, which would it be?

L: If I had to choose just one?

U: Yeah.

L: I don't know. That's really interesting. Which theory do I really believe? I guess I've always been uncertain which to believe and it's been there in the background you know, but I don't know. I guess if I had to choose, if I have to choose one, I believe it's a mathematical trick, a mathematician's game. ....What

possible reason could there be for height, ...for nature, to follow some weird bizarro function?

The above was a short section of the first probability interview I conducted . Until the last exchange transcribed, I had the feeling that the interview was not very interesting for Lynn. But after that last exchange she got very animated and involved in the interview. This question of the reality vs. "trickiness" of the normal and other distributions occupied her for much of the next few discussions. She came back to it again and again.

At the time, I was a bit surprised by this reaction. Earlier in the interview, when I asked her about infinitely many height discriminations, I think I was trying to steer Lynn into making a distinction between a discrete and continuous distribution. This, I thought, might have resolved her dilemma about ignoring the "bump in the middle". But I was not expecting to hear such basic doubts about the validity of statistics from someone with so much experience studying and using statistical methods .

Lynn's question was, as she says, "in the background" all the time when she thought about statistics. How could it not be? At the same time that she was solving probability and statistics problems for homework, and later on when she actually implemented a study which used statistical tests to evaluate treatments, she was never really sure that the formulae she was using had any explanatory power, any real meaning other than mathematical conventions. Is it any wonder, then, that she was engaged by this question? Up until now, no one had engaged her in discussion about how to interpret the mathematical formulae she was using. She was left feeling that her own work played by the "official rules" but in a way was not accessible to her -- and she harbored doubts about its validity. Her questions, while essentially epistemological, go to the heart of the mathematics of probability. Note that, in emphasizing the normal distribution, her courses had led her to see the process of describing height by a curve as a single process with a single agent (possibly the mathematician) controlling it. The idea of the height distribution as being emergent from many interacting probabilistic factors was not in awareness.

Had we tested Lynn on textbook statistics problems, she would have performed flawlessly . Yet, even in a interviewee of such sophisticated background, epistemological anxiety was lurking in the background. Lynn knew that she did not really know what gave validity to the procedures she had so laboriously mastered. Through engaging in this interview, her fundamental confusion is exposed. To an uninformed observer, it might seem that the interview has caused Lynn to regress, confusing her about material she has mastered. But, in fact, by expressing her partial understandings of distributions through asking whether normal distributions are "just a mathematician's trick", Lynn takes a big step towards a more Connected Mathematical understanding of distributions.

In the next interview, we will see how another interviewee, engaged in building a StarLogo model of normal distributions, deals with these same issues.

6.2 Modeling Normal Behavior - Alan's Hopping Rabbits

Alan, a graduate student in media studies had a strong college mathematics background. Although, he is mathematically quite sophisticated, Alan also expresses confusion about normal distributions:

A: I never really understood normal distributions. I mean I can do the problems, I've even helped Wendy [his wife] with her statistics problems, but what is really going on? Why should height fall into a normal distribution?

As the interview progressed, Alan became intrigued by the question: "why is height normally distributed?" After some prodding by the interviewer to come up with an answer to his question, Alan made a conjecture as to why height is normally distributed:

A: You start out with this one couple, Adam and Eve say, and they're a certain height. No, make this simpler, we just have Adam and he's a certain height. Now let's suppose we have parthenogenesis, (is that the word?) and Adam has kids on his own. And suppose his kids are a bit off from where he is [that is, slightly different heights] due to copying errors or something. Then they sort of form a little bell curve - anyway a binomial distribution. Now suppose they have children and that process continues many generations, then wouldn't you wind up with a normally distributed population?

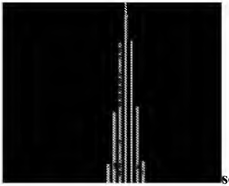
To explore this conceptual model, Alan decided to write a StarLogo model with me. The model is initialized to have thousands of "rabbits" at the bottom middle of the graphics screen. (Call the initial location of the rabbits the "origin". For reference, a green vertical line is placed at the origin.) Rabbits can hop right or left from the origin on a line at the bottom of the screen. (Call it the hopping line) Each rabbit has a set of possible step sizes which it can hop in one clock "tick". Associated with each step size is a probability - the probability that the rabbit will choose that step size. To make the interface easier for himself and other possible users who might not be comfortable with probabilities (expressed as real numbers between 0 and 1), the probabilities of rabbit steps are expressed as integer relative odds. The rabbits are initialized with the command "setup-rabbits" which takes three arguments: the number of rabbits, a list of step-sizes and a list of probability-ratios. So, for example, if one wanted to create 1000 rabbits that could hop to the right 2 units with probability 3/4 and hop to the left 1 unit with probability 1/4, one would type the command

setup-rabbits 1000 [2 -1] [3 1].

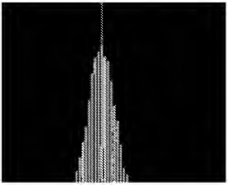
After executing this command, the rabbits would be all piled on top of each other at the middle bottom of the screen. It is then possible to perform experiments and let them hop for a while. At each clock cycle each rabbit picks a step to hop according to the probabilities and hops that distance on the hopping line. Once the rabbits have hopped for one clock tick, they are no longer all in the same place. They are still on the hopping line, but at different x positions on that line. Above each position at which there are rabbits present, we display a yellow column - the height of which is proportional to the number of rabbits at that location. In this way an evolving "living" histogram of the distribution of rabbit locations on the hopping line appears on the screen.

To make this a bit clearer, letês take a simple example:

If we initialize the rabbits with the command setup-rabbits 8000 [1 -1] [1 1], then 8000 rabbits will appear at the origin. Each rabbit has been initialized so that it can only hop either one unit to the left or one unit to the right with equal probability. If we let them hop one step, then because the probability of moving left is the same as the probability of moving right, approximately half the rabbits will move one unit left and another half will move one unit right. As a result two roughly equally high yellow columns will appear on the screen above locations x=1 and x=-1. If we now let the rabbits hop one more step, there will be a tall yellow column in the middle and shorter, roughly equally high, yellow columns to the right and to the left. Continuing this process will lead to histograms of binomial distributions symmetric about the origin. (See figure below).



setup-rabbits 8000 [1 -1] [1 1] after 10 "hops"



setup-rabbits 8000 [1 -1] [1 1] after 40 "hops"

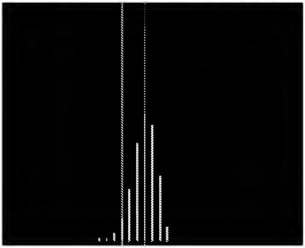
In the above simple example, when the rabbits hop, their average location does not move from the origin ã the histogram of rabbits is symmetric about the origin. To help visualize the movement of the rabbits when the average location does change, a purple column is displayed above the average location and a green column "remembers" where the rabbits started. Natural questions that arise in this rabbit jumping "microworld", then, are: what will cause the average location of the rabbits to change and what will cause more of them to be one side of the average than on the other?

As you may recall, the reason Alan and I created this model was to investigate Alan's theory of height distributions. One way to think of the rabbits when they're all at the origin is as a population of individuals all of a standard height. When rabbits take a step they represent new individuals in the population who have deviated slightly from their parents' height. In our simple example the average height of the population does not change and the population heights are symmetric about this average.

When viewing this example, Alan exclaimed: "I see why population heights are symmetrical, the rabbits hop the same distance to the right as to the left". He then typed in the command setup-rabbits 8000 [2 -1] [1 1]. "Now", he said, "the graph won't be symmetrical, it will skew to the right". But to Alan's surprise, when the rabbits hopped, the histogram remained symmetrical, but the purple line representing the average rabbit location (or average population height) moved to the right.

Alan quickly understood the "bug" in his thinking:

"I see. Rabbits hop farther to the right when they hop but they're still just as likely to hop right as left. So, the distribution will remain symmetric, it will just move to the right one unit on average for each step. To make the distribution asymmetric, you need to change the probabilities not the step sizes."


setup 8000 [2 -2] [3 1] after 10 hops

From the above experiment, Alan concluded:

If children are just as likely to be shorter as they are taller than their parents but when they are taller their difference from their parents is larger, then the population as a whole will slowly get taller. But since we know that height of the population as a whole is symmetrically distributed, children must be equally likely to be taller as they are to be shorter than their parents. I see why normal distributions are so common. Whenever we make a measurement, we're just as likely to make a mistake in one direction as the other, so the resultant distribution of measurements will be normal. Its average will be the true value of the measurement and the spread of the graph will depend on the accuracy of the measurement.

Alan then went on to investigate what would happen to the distributions if he gave the rabbits a much larger "palette" of step sizes. This investigation leads him to more insights about how different micro- level rules produce different probability distributions. When I asked Alan to sum up what he has learned from his

modeling experience, he expresses satisfaction that he has gained a deeper understanding of binomial distributions and "what kind of animal a distributions is". But, he also says that perhaps he has not yet "entirely understood normal distributions". He speculates that the normal distribution is "the limit of the binomial distribution as "n" approaches infinity -- when you fill in the gaps between the bars of the binomial". He worries, however, that he doesn"t yet sufficiently understand the connection between the binomial distribution and the formula he remembers for normal distributions: "how does the integral of e -x2/2 come out of this?".

It is important to remember that Alan began the interview with a command of elementary probability and statistics. He was facile with the mathematics of binomial processes and knew the standard parameters and statistics of normal distributions. It was the connection between these two areas of his knowledge that was missing for Alan. School mathematics, like pre-nineteenth century mathematics keeps these two disciplines separate. Indeed, with limited computational power, it is hard for a school curriculum to make a meaningful connection between the two. Object-based parallel modeling languages bring the two back together. By enabling Alan to give probabilistic rules to the individuals and see the resulting pattern of the ensemble, the StarLogo modeling language enabled him to make these connections. Though Alan has made significant progress in his understanding of distributions, he is not done. His desire to further connect his new understanding to the formula he learned in school exemplifies his appropriation of the Connected Mathematics approach.

7.0 Discussion

It is clear that the source of Lynn and Alan's difficulty with P&S did not lie in a lack of technical proficiency. By most standards, their understanding was advanced. The issue is that their understanding was not well connected. The case studies demonstrate their ability to make these connections and develop rich intuitions in areas, which, in the past, they accommodated by mechanically applying formulae. This supports the Connected Probability project's stance of eschewing the accomodationist view which says "that's the way it is" for P&S, and, instead, viewing P&S as fitting within the larger framework of research in understanding mathematics. Development of intuition and changes in understanding can happen in ways consistent with a broad developmentalist stance. In this discussion, I emphasize the features of the Connected Mathematics therapy that facilitated the development of effective intuitions about normal distributions.

Both Lynn and Alan had strong mathematical backgrounds, had done graduate coursework in mathematics and had excelled. Both of them were very comfortable doing the formal manipulations that constitute the major component of traditional instruction in P&S. Nonetheless, both explicitly expressed considerable discomfort concerning the notion of normal distribution ã the most salient example of the core statistical notion of probability distribution. Regarding the nature of normal distributions, they suffered from epistemological anxiety. The therapy provided in these learning interviews consisted of two main interventions: 1) Validation of the importance of Lynn and Alan's mathematical voice, their personal view of the mathematics and its relation to their own experience. This validation was achieved by explicit avowal at the start of the interview, by open-ended interviews that followed the interviewee's own thinking and by long drawn out interviews that declared of themselves that the interviewee's conceptions were valuable and valued. 2) The use of an object-based parallel modeling language, StarLogo, in which to build computational models of probabilistic phenomena and successively refine them. These supports provided a context in which to work through their epistemological anxiety.

Though, clearly, neither Lynn nor Alan were experts in P&S, their sophisticated mathematics background as well as their formal mastery of P&S techniques would qualify them as experts in the eyes of many educators. These "near-experts" were no less confused about normal distributions than were the novice interviewees. Indeed, instructors of P&S with whom I have shared Lynn's interview have been "appalled at the incoherence of her remarks ", yet, instructors like these, awarded top grades to Lynn in her P&S courses. Interviews such

as these cast serious doubt on theories which assert a strong disjunction between novice and expert understanding. Even experts must laboriously map out clarity from a very messy "terrain". By overemphasizing formal operational thought, typical mathematics education allows intellectual accommodation to be an adequate endpoint. In Connected Mathematics, understanding is taken to mean "multiply connected" and not cleanly and uniquely specified. This embraces "messy" conceptions at all levels of expertise in the process of learning. One has never "arrived".

Rather than emphasizing problem solving, Connected Mathematics emphasizes problem posing. In problem solving, the questions are often disembodied and not the learnerês own. Within Connected Mathematics, modeling is both a medium in which learners can formulate their questions in a precise way and a method for building their understanding.

Once Alan posed the problem of the symmetry of the distribution, his reasoning is expressed and refined in interaction with the model he has built. Alan strives to find the connections between his representation of the individual rabbits behaviors and the global pattern that they form. He debugs his model by varying each of the three inputs to the run-rabbits procedure, focusing particularly on the lists of steps and ratios. How will variations in the lengths of the lists and the relative magnitudes of the steps and ratios affect this emergent pattern? He alternates his focus from the micro- level, the behavior of the individual rabbits to the macro-level of the distribution. To better keep track of the change in the global pattern over time, he takes advantage of StarLogoês open-ended programmability and builds representational aids such as the "origin line", the "hopping line" and color-coded classes of rabbits. These representations and tools allow him to debug his original conception that the relative size of the steps is the key to an asymmetric distribution and, instead, to see the relative size of the probabilities as the key factor. By building distributions with probabilistic parts and making them work, Alan makes probability ratios as concrete for himself as step-sizes, and sees how distributions are built out of these concrete building blocks. Extending, enhancing and debugging the computational model itself is essential to the activity of modeling. This distinguishes it from model consumption in which learners are left to tweak the inputs of an expert's model.

In the Connected Probability project, probability distributions are viewed as a kind of emergent phenomena. Rather than manipulate the tokens of formalism, learners are invited to negotiate the interaction of the micro- and macro- levels of probabilistic and statistical phenomena. Through connecting these levels of emergent phenomena, learners understand the "mechanisms" of probability distributions -- how they are assembled from their constituent elements. When these interactions are understood, the global patterns become more than just descriptions to be memorized, they are tools to be used.

Binomial probability distributions are elementary examples of emergent phenomena -- emerging from the behavior of non-interacting individuals. Other probability distributions emerge from more interactive individual behavior. Many natural phenomena can be modeled as systems whose global behavior are not predictable using non-computational mathematics. Such phenomena as the shape of a snowflake, the dynamics of co-extenisve wolf and sheep populations, the economy of a country -- all of these can be modeled as emerging from the interactions of their constituent individuals. Learning to see emergent phenomena -- that is learning to describe the phenomena as arising from the interactions of distributed parts -- is, in itself, a worthy goal for educators. From this perspective, P&S becomes not an end, but an entry point to the world of complex systems.

In the Connected Probability project, the role of technology is as a medium of personal expression and articulation of ideas. Empowering literacy implies being able to say something of your own, rather than just manipulate the texts of others. While models are ubiquitous in the current learning literature, modeling is not. Modeling calls on learners to say something on their own, to be authors of mathematics. This implies the best use of technology is for learners to be model builders not just model consumers. The StarLogo modeling language has specific features which allow for this articulation:

Modeling at the level of objects: StarLogo allowed Alan to create, visualize and modify individual rabbits. Alan was, thus, able to experiment with individual rabbits whose hopping behavior he could track, control, model with his own body. These features of the language enabled Alan to leverage his knowledge of individual behavior to knowledge of the ensemble ã to connect the micro- with the macro-.

Parallelism: The ability to control thousands of rabbits at once gave Alan the experimental apparatus he needed to conduct his investigation in "real time". The ability to experiment with collections of rabbits, to enable sub-populations of rabbits to have different behaviors was crucial to his construction of the distribution.

General purpose programmability: The ability to create his own representations (as opposed to manipulating the parameters of a pre-conceived model) gave Alan the freedom to construct normal distributions in a way that arose from his own nascent understanding of them. It also enabled him to create computationally active tools, such as the "hopping line". These refinements of his computational model provided greater support for his experimentation and, in turn, led to a refinement of his mental model.

In a Connected Mathematics learning environment, learners are supported in actively connecting areas of their knowledge that have, hitherto, remained separate. Computational platforms, in their ability to bring into being new forms of representation and symbolization, can be powerful tools for making these connections. Object-based parallel modeling provides a new way of coming to understand and articulate understandings of probability and statistics and, in so doing, engage and relieve epistemological anxiety.

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